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BRST cohomologies of non-critical RNS strings

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Abstract

The BRST cohomologies of non-critical massive RNS theories are analysed in detail in the range of dimensions $1 < d < 10$. It is shown that the spaces of physical states admit the relative BRST resolutions: the theorems on vanishing of bigraded-Dolbeaut type cohomologies and relative BRST cohomologies are proved. The no-ghost theorem for relative classes guarantees quantum mechanical consistency of these string models. The explicit correspondence between relative cohomology states and the physical states of 'old covariant' formalism is established: a light-cone gauge slice is shown to be a local section of relative cohomologies. The problem of reconstruction of absolute BRST cohomologies out of those of the relative complex is also explicitly solved.

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Introduction

This paper reports on a natural continuation of the research on non-critical bosonic strings recently published in [1]. It is devoted to a detailed study of quantum BRST cohomologies of non-critical fermionic string models of the Neveu–Schwarz and Ramond types. These models are, as shown in [2], gauge equivalent on the quantum level to the non-critical spinning string theories [3] with longitudinal degrees of freedom [4].

It is hopeless to expect that the string models considered in this paper have something in common with dimensionally reduced critical string theories. From the purely theoretical point of view, they seem to provide a relativistic invariant and tractable quantum description of one-dimensional extended objects in subcritical dimensions. The main physical motivation for string models in subcritical dimensions goes back to an old and attractive idea that low-energy hadronic physics should be described by some effective theory of string type. The construction of a consistent theory of string interactions would be a step towards verification of this idea. Despite the remarkable structural similarities of free non-critical string models to those in $d = 10$ or 26, the mechanism of joining–splitting [5] interactions cannot be extended to subcritical dimensions, at least in its pure form. The fundamental interaction vertices are simply not Lorentz covariant. The methods of conventional conformal field theory [6, 7] seem to have a restricted area of application too. One important reason is that the ground states of non-critical bosonic strings as well as non-critical fermionic strings are not $Sl(2;\mathbb{R})$ invariant and fall into modules of subsidiary continuous series of the representations of this group [8]. Although these representations have 'stringy' realizations in the spaces of functions on the unit circle, the corresponding scalar products are highly non-local.

On the other hand, the mass and spin spectra of these non-critical string models look promising from the point of view of their hadronic interpretation [9, 10]: there are no massless excited states, tachyonic ground states can be eliminated from the Neveu–Schwarz spectrum by consistent projection of GSO type [11] and the resulting theory is by no means supersymmetric. It is not clear, however, that the ground tachyons should be necessarily eliminated from the spectrum of a consistent theory. It may quite well be that their presence generates a mechanism which allows one to look at the quantum string states as the states of a confined system which are only locally visible (for example, in local light-cone frame) as free particles. It is then natural to bring the scalar nature of the ground tachyons into question too. It may well be that globalization of the light-cone description of non-critical strings demands putting the ground state into a non-trivial representation of the little group of its momentum ($SO(1,2) \overset{\text{loc}}{\sim} Sl(2,\mathbb{R})$ in $d = 4$).

In order to pursue these questions, with the problem of consistent interaction theory being a prominent one, a better understanding of the quantum geometry of non-critical strings is necessary. The research presented in this paper is a small step towards this ultimate goal.

Although the theory of critical string interactions was originally formulated within the framework of 'old covariant' formalism or in terms of Mandelstam light-cone diagrams, it achieved its final form within the framework of the BRST formulation [12]. Soon after the invention of this formalism, the critical bosonic stings [13] and critical RNS models [14–17] were shown to admit a consistent BRST description. The first attempt to apply BRST methods to non-critical strings was presented in [18]: it was shown that there exists a quantum complex corresponding to the canonical formulation [19] of Polyakov theory, provided that the Liouville coupling constant and intercept parameter take their critical values. It is not difficult to foresee an analogous result in the case of non-critical fermionic strings. The content of the corresponding cohomology space is less obvious.

The cohomologies of BRST complexes corresponding to non-critical RNS models are investigated in this paper. These models can be obtained by canonical quantization of the systems defined by covariant spinning string action [3], supplemented by supersymmetric Liouville action [20]. As their bosonic counterpart [22], the non-critical RNS strings are described in the flat superconformal gauge [21] as canonical classical systems with constraints of mixed type [2].

The standard BRST complexes used in this paper are, from the mathematical point of view, nothing other than equivariant-type [23] complexes of super-Virasoro algebras with values in the space of first quantized non-critical RNS models. (It should be stressed, however, that the constraints of mixed type are treated here as if they were of first class.)

The paper is organized as follows. The first quantized non-critical RNS models are defined in section 1, followed by the corresponding BRST complexes. Special attention is devoted to a detailed description of the relative complex and bigraded complexes of Dolbeaut type [24]. These last serve as very effective tools in identification of relative cohomology classes, but may also be important for formulation of gauge string theory [25]. The vanishing theorems for bigraded and relative cohomologies are formulated and proved in section 2: the approach is based on a remarkably simplified version of the technology of spectral sequences [28, 29] and provides a transparent relation [26, 27] between the kinematical situation and the vanishing of cohomologies. The absolute cohomology spaces are explicitly reconstructed out of relative classes. A proof of the no-ghost theorem based on the ideas of [16, 30] is given in section 3. There is also some space devoted to the problem of identification of relative cohomology classes as physical states of the 'old covariant' formalism. Finally, the GSO projected relative complexes are defined and the content of their cohomologies is briefly discussed.

1. Massive fermionic strings and BRST complexes

First-quantized massive (non-critical) fermionic string models are described in pseudo-unitary spaces $\mathcal{F}_\varepsilon$ using the algebra generated by the elementary d-dimensional string modes:

$$\begin{array}{ll} [a^\mu_m, a^\nu_n] = m\eta^{\mu\nu}\delta_{m+n} & m, n \in \mathbb{Z} \\ \{d^\mu_r, d^\nu_s\} = \delta_{r+s}\eta^{\mu\nu} & r, s \in \mathbb{Z} + \frac{\varepsilon}{2} \quad \varepsilon = 0, 1 \quad 0 \leqslant \mu \quad \nu \leqslant d-1 \\ [a^\mu_0, q^\nu_0] = -\mathrm{i}\eta^{\mu\nu} & \end{array} \tag{1}$$

and an additional family of operators

$$[u_m, u_n] = m\delta_{m+n} \qquad \{t_r, t_s\} = \delta_{r+s} \qquad m, n \in \mathbb{Z} \quad r, s \in \mathbb{Z} + \frac{\varepsilon}{2} \tag{2}$$

to describe the bosonic and fermionic Liouville excitations. The parameter ε is introduced to distinguish between Ramond ($\varepsilon = 0$) and Neveu–Schwarz ($\varepsilon = 1$) boundary conditions. The dimensionless bosonic zero modes a^μ_0, q^ν_0 are related with centre-of-mass coordinate $\sqrt{\alpha}x^\nu = q^\nu_0$ and its canonically conjugated momentum $P^\mu = \sqrt{\alpha}a^\mu_0$. The Liouville zero mode is set to take an arbitrary but fixed real value $u_0 \equiv \varrho$. (The consistency conditions of the classical variational problem demands [2, 22] $\varrho = 0$.)

The representation space for (1) and (2) is constructed as the direct integral of the pseudo-unitary spaces $\mathcal{F}_\varepsilon(p)$

$$\mathcal{F}_\varepsilon = \int_{\mathbb{R}^d} d^d p\, \mathcal{F}_\varepsilon(p) \tag{3}$$

over d-dimensional spectrum of momentum operators P^μ. Every $\mathcal{F}_\varepsilon(p)$ is constructed to carry the representation of the subalgebra generated by all elementary string and Liouville operators (1), (2) except of bosonic zero modes. The constructions of $\mathcal{F}_1(p)$ and $\mathcal{F}_0(p)$ are slightly different due to the presence of Clifford generators in the Ramond sector.

In the case of Neveu–Schwarz boundary conditions the space $\mathcal{F}_1(p)$ is the Fock module built up over the unique vacuum vector $\omega(p)$, which is assumed to be the eigenstate of momentum operators $P^\mu\,\omega(p) = p^\mu\,\omega(p)$, and satisfies

$$a^\mu_m\omega(p) = d^\mu_r\omega(p) = u_m\omega(p) = t_r\omega(p) = 0 \qquad m, r > 0. \tag{4}$$

The scalar product in $\mathcal{F}_1(p)$ is fixed by supplementing (1), (2) and (4) by formal conjugation rules

$$(a^\mu_m)^* = a^\mu_{-m} \qquad (d^\mu_r)^* = d^\mu_{-r} \qquad (u_m)^* = u_{-m} \qquad (t_r)^* = t_{-r} \tag{5}$$

and by imposing the normalization condition on the vacuum vectors $(\omega(p), \omega(p')) = \delta(p' - p)$.

It is convenient to introduce here the NS fermion parity operator

$$(-1)^{F_1} := \exp \mathrm{i}\pi \sum_{r>0} (d_{-r} \cdot d_r + t_{-r}\, t_r). \tag{6}$$

It will be necessary for further constructions of graded tensor products.

The structure of the Ramond excitation space $\mathcal{H}_0(p)$ is a little more complicated due to the presence of fermionic zero modes d^μ_0, t_0, which generate the real Clifford algebra $\mathcal{C}(d, 1)$

of $(d+1)$-dimensional space with Lorenzian metric. If, in addition, one requires a well defined fermion parity operator (an analogue of that of (6)—anticommuting with all fermionic modes) the extension of the zero-mode sector by an additional generator Γ^f is inevitable. The space $\mathcal{F}_0(p)$ must carry the representation of the real Clifford algebra $\mathcal{C}(d+1,1)$ and can be constructed as the graded tensor product

$$\mathcal{F}_0(p) = \tilde{\mathcal{F}}(p) \otimes_{\mathbb{Z}_2} \mathcal{S}(d+1,1) \tag{7}$$

of the Clifford algebra module $\mathcal{S}(d+1,1)$ with the auxiliary Fock space $\tilde{\mathcal{F}}(p)$. (It is assumed that $\mathcal{S}(d+1,1)$ is an irreducible module for complexified algebra. The grading in $\mathcal{S}(d+1,1)$ is understood as the decomposition into the sum of ± 1 eigensubspaces of Γ^f. More information on Clifford modules may be found in the appendix A.) The space $\tilde{\mathcal{F}}(p)$ is generated out of the unique vacuum vector $\tilde{\omega}(p)$ by the set of non-zero modes $\tilde{a}^\mu_m, \tilde{u}_m, \tilde{d}^\mu_r, \tilde{t}_r$ with $m, r \in \mathbb{Z} \setminus \{0\}$. It is assumed that they satisfy the same (anti)commutation relations as those of (1), (2) and annihilate the vacuum $\tilde{\omega}(p)$ according to the rule introduced in (4).

The generators of the Ramond algebra are realized on $\mathcal{F}_0(p)$ by the following set of operators:

$$\begin{aligned} &a^\mu_m = \tilde{a}^\mu_m \otimes 1 \qquad u_m = \tilde{u}_m \otimes 1 \qquad d^\mu_r = \tilde{d}^\mu_r \otimes \Gamma^f \qquad t_r = \tilde{t}_r \otimes \Gamma^f \qquad m, r \neq 0 \\ &d^\mu_0 = 1 \otimes \frac{1}{\sqrt{2}}\Gamma^\mu \qquad t_0 = 1 \otimes \frac{1}{\sqrt{2}}\Gamma^l. \end{aligned} \tag{8}$$

The matrices $\{\Gamma^\mu, \Gamma^l, \Gamma^f\}$ represent the canonical generators of Clifford algebra as endomorphisms of $\mathcal{S}(d+1,1)$.

The fermion parity operator in $\mathcal{F}_0(p)$ with all the desired properties is then defined by

$$(-1)^{F_0} := \exp \mathrm{i}\pi \tilde{F} \otimes \Gamma^f \qquad \tilde{F} = \sum_{r>0} \tilde{d}_{-r} \cdot \tilde{d}_r + \sum_{r>0} \tilde{t}_{-r}\tilde{t}_r. \tag{9}$$

In order to introduce the scalar product in the Ramond sector one imposes on the tilded factors of the operators of (8) the same conjugation rules as those of NS space (5). The scalar product of the Ramond vacuum vectors $u(p) = \tilde{\omega}(p) \otimes u$; $u \in \mathcal{S}(d+1,1)$ is defined by $(u(p), u'(p')) := \delta(p'-p)\langle u, u'\rangle$, where $\langle\,,\rangle$ denotes a Hermitian pairing (A.3) on the Clifford module such that all Γ-matrices are $\langle\,,\rangle$-skew symmetric. This gives the following conjugation properties of Ramond modes (8):

$$(a^\mu_m)^* = a^\mu_{-m} \qquad (d^\mu_r)^* = -d^\mu_{-r} \qquad (u_m)^* = u_{-m} \qquad (t_r)^* = -t_{-r} \tag{10}$$

with respect to the resulting scalar product on $\mathcal{F}_0(p)$.

The constraint operators on $\mathcal{F}_\varepsilon$ are given by the standard normally ordered expressions:

$$\begin{aligned} L_m &= \tfrac{1}{2}\sum_{n\in\mathbb{Z}} : a_{-n} \cdot a_{n+m} : + \tfrac{1}{2}\sum_{r\in\mathbb{Z}+\frac{\varepsilon}{2}} r : d_{-r} \cdot d_{r+m} : \\ &\qquad + \tfrac{1}{2}\sum_{n\in\mathbb{Z}} : u_{-n}u_{n+m} : + 2\mathrm{i}\sqrt{\beta} m u_m + (2\beta - a_\varepsilon)\delta_{m0} + \tfrac{1}{2}\sum_{r\in\mathbb{Z}+\frac{\varepsilon}{2}} r : t_{-r}t_{r+m} : \\ G_r &= \sum_{n\in\mathbb{Z}} a_{-n} \cdot d_{n+r} + \sum_{n\in\mathbb{Z}} u_{-n}t_{n+r} + 4\mathrm{i}\sqrt{\beta} r t_r. \end{aligned} \tag{11}$$

The real Liouville coupling constant β and a real number a_ε, defining the beginning of the mass spectrum of the physical states, are at the moment left as the free parameters of the quantum model. They will be fixed at their critical values by the requirement of the existence of BRST complex corresponding to the system (11) of constraints.

The structural relations of (11) reads

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \tfrac{1}{8}c(m^3 - \varepsilon m)\delta_{m+n} + 2ma_\varepsilon\delta_{m+n} \\ [L_m, G_r] &= \left(\frac{m}{2} - r\right) G_{m+r} \\ \{G_r, G_s\} &= 2L_{r+s} + \tfrac{1}{2}c(r^2 - \varepsilon\tfrac{1}{4})\delta_{r+s} + 2a_\varepsilon\delta_{r+s} \end{aligned} \tag{12}$$

with the value of central charge being given by $c = d + 1 + 32\beta$.

It is worth recalling that L_0 as well as the level operator $R = \sum_{n>0}(a_{-n} \cdot a_n + u_{-n}u_n) + \sum_{r>0} r(d_{-r} \cdot d_r + t_{-r}t_r)$ are diagonalizable. The set of eigenvalues of R is given by $\mathrm{spec}(R) = \mathbb{N}_\varepsilon = \mathbb{N}(1 - \frac{\varepsilon}{2}) \cup \{0\}$, where $\mathbb{N}$ denotes the set of positive integers.

Every space $\mathcal{F}_\varepsilon(p)$ of (3) decomposes into a direct sum of finite-dimensional subspaces of fixed non-negative level number. The dimensions of the corresponding eigensubspaces are given in terms of standard bosonic (−) and fermionic (+) partition functions $P_{\varepsilon\mp}(q) = \prod_{n>0}(1 \mp q^{n-\frac{\varepsilon}{2}})^{\mp 1}$:

$$\mathcal{F}_\varepsilon(p) = \bigoplus_{\mathrm{N}\in\mathbb{N}_\varepsilon} \mathcal{F}_\varepsilon^N(p) \qquad \dim \mathcal{F}_\varepsilon^N(p) = q^{-N}\, 2^{(1-\varepsilon)[\frac{d+2}{2}]} P_-(q)^{d+1}\, P_{\varepsilon+}(q)^{d+1}|_0 \tag{13}$$

where $[\cdot]$ is the integer part of a number and $|_0$ denotes the constant term of the series. The coefficient in front of the dimension formula (13) counts the degeneracy of the vacuum state in the Ramond case and is equal to $\dim \mathcal{S}(d+1, 1)$.

The decomposition of $\mathcal{F}_\varepsilon(p)$ into a direct sum of finite-dimensional subspaces of fixed level allows one to be more precise about the structure of the space $\mathcal{F}_\varepsilon$. It is assumed that the elements of this space have only a finite number of non-zero components in the fixed level subspaces of this decomposition.

The subspace of physical states of 'old covariant' formulation $\mathcal{F}_\varepsilon^{\mathrm{phys}} \subset \mathcal{F}_\varepsilon$ is defined as the set of vectors Ψ satisfying

$$\mathcal{F}_\varepsilon^{\mathrm{phys}} = \left\{\Psi;\, L_n\Psi = G_r\Psi = 0;\, n \geqslant 0, r \geqslant \frac{\varepsilon}{2}\right\}. \tag{14}$$

The conditions above admit a slight generalization [2] leading one to a description of continuous and discrete series of unitary string models with longitudinal degrees of freedom. However, the generalized models do not admit the BRST resolution, and for this reason they are not considered here.

In order to construct the BRST complex associated with the constraints (11), (12) one introduces the corresponding ghost sector. The ghost sector is defined as the representation space $\mathcal{C}_\varepsilon$ of ghost (anti)commutation relations

$$\{b_n, c_m\} = \delta_{m+n} \qquad [\beta_r, \gamma_s] = \delta_{r+s} \qquad m, n \in \mathbb{Z} \quad r, s \in \mathbb{Z} + \frac{\varepsilon}{2}. \tag{15}$$

The space $\mathcal{C}_1$ corresponding to the Neveu–Schwarz algebra is generated out of the unique vacuum state ω satisfying

$$\begin{aligned} &b_n\omega = 0 \qquad c_m\omega = 0 \qquad n \geqslant 0 \quad m > 0 \\ &\beta_r\omega = 0 \qquad \gamma_s\,\omega = 0 \qquad r > 0 \quad s > 0. \end{aligned} \tag{16}$$

The ghost space $\mathcal{C}_0$ of the Ramond case stems from two independent vacuum vectors ω_λ; $\lambda = 0, 1$ (this 'picture' [7] for the representation of bosonic ghost commutation relations is the only one [17] consistent with natural ghost conjugation rules and which guarantees the spectrum of energy being bounded from below) defined to satisfy the following conditions:

$$\begin{aligned} &b_n\omega_\lambda = 0 \qquad c_m\omega_\lambda = 0 \qquad n \geqslant 0 \quad m > 0 \\ &\beta_r\omega_\lambda = 0 \qquad \gamma_s\,\omega_\lambda = 0 \qquad r \geqslant \lambda \quad s \geqslant 1 - \lambda. \end{aligned} \tag{17}$$

The non-degenerate scalar product in $\mathcal{C}_\varepsilon$ is fixed by imposing the canonical conjugation properties on the ghost modes

$$(c_m)^* = c_{-m} \qquad (b_m)^* = b_{-m} \qquad (\gamma_r)^* = \gamma_{-r} \qquad (\beta_r)^* = -\beta_{-r} \tag{18}$$

and assuming the normalization conditions $(\omega, c_0\,\omega) = 1$ for the Neveu–Schwarz vacuum and $(\omega_\lambda, c_0\,\omega_{\lambda'}) = 1 - \delta_{\lambda\lambda'}$ for Ramond ground states respectively.

The spaces $\mathcal{C}_\varepsilon$ are graded by the eigenvalues of ghost number operators:

$$\mathrm{gh}_\varepsilon = \sum_{n\in\mathbb{Z}} : c_{-n} b_n : + \sum_{r\in\mathbb{Z}+\frac{\varepsilon}{2}} : \gamma_{-r}\beta_r : + \tfrac{1}{2}(c_0 b_0 - b_0 c_0) + (1-\varepsilon)\tfrac{1}{2}(\gamma_0\beta_0 + \beta_0\gamma_0) \tag{19}$$

such that $\mathrm{gh}_1\, \omega = -\frac{1}{2}\,\omega$ and $\mathrm{gh}_0\, \omega_\lambda = -\lambda\,\omega_\lambda$. According to these conventions the ghost numbers are integral or half-integral depending on the sector: $\mathrm{spec}\,(\mathrm{gh}_\varepsilon) = \mathbb{Z} + \frac{\varepsilon}{2}$.

The realization of superconformal algebra in $\mathcal{C}_\varepsilon$ is given by the operators

$$\begin{aligned} \mathcal{L}_m &= \sum_{n\in\mathbb{Z}} (n-m) : c_{-n} b_{n+m} : + \sum_{r\in\mathbb{Z}+\frac{\varepsilon}{2}} (r - \tfrac{1}{2}m) : \gamma_{-r}\beta_{r+m} : \\ \mathcal{G}_s &= -2 \sum_{r\in\mathbb{Z}+\frac{\varepsilon}{2}} : \gamma_{-r} b_{r+s} : + \sum_{n\in\mathbb{Z}} (s - \tfrac{1}{2}n) : c_{-n}\beta_{n+s} : \end{aligned} \tag{20}$$

which are normally ordered with respect to (16) and (17) respectively. They satisfy the structural relations of (12) with the value of central charge $c = -10$ and $a_\varepsilon = \frac{\varepsilon}{2}$.

The differential in $\mathcal{C}_\varepsilon$ is defined by

$$d = \tfrac{1}{2}\sum_{m\geqslant 0} c_{-m}\mathcal{L}_m + \tfrac{1}{2}\sum_{m>0} \mathcal{L}_{-m} c_m + \tfrac{1}{2}\sum_{r\geqslant 0} \gamma_{-r}\mathcal{G}_r + \tfrac{1}{2}\sum_{r>0} \mathcal{G}_{-r}\gamma_r. \tag{21}$$

It neither commutes with the operators (20) nor it is nilpotent:

$$d^2 = -\sum_{n>0}\left(\frac{10}{8}(n^3 - \varepsilon n) + \varepsilon n\right) c_{-n} c_n - \sum_{r>0}\left(\frac{10}{2}\left(r^2 - \frac{\varepsilon}{4}\right) + \varepsilon\right)\gamma_{-r}\gamma_r. \tag{22}$$

The total space $\mathcal{C}(\mathcal{F}_\varepsilon)$ of the string BRST complex is defined as $\mathbb{Z}_2$ graded tensor product:

$$\mathcal{C}(\mathcal{F}_\varepsilon) = \int_{\mathbb{R}^d} d^d p\, \mathcal{C}(\mathcal{F}_\varepsilon)(p) \qquad \mathcal{C}(\mathcal{F}_\varepsilon)(p) := \mathcal{F}_\varepsilon(p) \otimes_{\mathbb{Z}_2} \mathcal{C}_\varepsilon. \tag{23}$$

In order to respect the graded structure on the level of the algebras of elementary string and ghost modes, the fermionic ghost operators are replaced by $c_n \mapsto (-1)^{F_\varepsilon} \otimes c_n$ and $b_n \mapsto (-1)^{F_\varepsilon} \otimes b_n$. Their conjugation properties with respect to the canonical pairing on the tensor product (23) are changed accordingly: due to the presence of the skew Hermitian matrix Γ^f (9) in the fermion parity operator of the Ramond sector one has $(c_n)^* = (-1)^{1-\varepsilon} c_{-n}$ and $(b_n)^* = (-1)^{1-\varepsilon} b_{-n}$.

Every space $\mathcal{C}(\mathcal{F}_\varepsilon)(p)$ (as well as $\mathcal{C}(\mathcal{F}_\varepsilon)$) inherits the ghost number (19) gradation

$$\mathcal{C}(\mathcal{F}_\varepsilon)(p) = \bigoplus_{\kappa\in\mathbb{Z}+\frac{\varepsilon}{2}} \mathcal{C}^\kappa(\mathcal{F}_\varepsilon)(p). \tag{24}$$

The BRST operator is a differential in $\mathcal{C}(\mathcal{F}_\varepsilon)$ of ghost degree +1 and is defined according to a general prescription of [23]:

$$D : \mathcal{C}^\kappa(\mathcal{F}_\varepsilon)(p) \to \mathcal{C}^{\kappa+1}(\mathcal{F}_\varepsilon)(p) \qquad D = \sum_{n\in\mathbb{Z}} L_n c_{-n} + \sum_{r\in\mathbb{Z}+\frac{\varepsilon}{2}} G_r\gamma_r + d \tag{25}$$

where L_n, G_r are string constraint operators (11) and d is that of (21). The conjugation properties of D depend on the sector: $D^* = (-1)^{1-\varepsilon} D$.

From (12) and (22) it follows that the operator D is nilpotent provided the free parameters β and a_ε of the string model take their critical values:

$$\beta = \frac{9-d}{32} \qquad a_\varepsilon = \frac{\varepsilon}{2}. \tag{26}$$

The absolute cohomology spaces of the BRST complex are defined in the standard way:

$$\begin{aligned} &H^\kappa(\mathcal{F}_\varepsilon)(p) = \frac{Z^\kappa(\mathcal{F}_\varepsilon)(p)}{B^\kappa(\mathcal{F}_\varepsilon)(p)} \\ &Z^\kappa(\mathcal{F}_\varepsilon)(p) = \ker D|_{\mathcal{C}^\kappa(\mathcal{F}_\varepsilon)(p)} \qquad B^\kappa(\mathcal{F}_\varepsilon)(p) = \mathrm{Im}\, D|_{\mathcal{C}^{\kappa-1}(\mathcal{F}_\varepsilon)(p)}. \end{aligned} \tag{27}$$

The common convention of denoting the spaces of co-cycles of any complex under consideration by the root letter Z and the spaces of coboundaries by the root letter B is used throughout this paper.

The cohomology spaces (27) are determined by the cohomologies of several subcomplexes of (23). The most important one is the so-called relative complex—its cohomology classes admit direct quantum mechanical interpretation, provided the no-ghost theorem can be proved. It will be demonstrated that they represent the physical states of a massive string.

It is necessary to recall some details of its structure. The construction of the relative complex is much more straightforward in Neveu–Schwarz than in the Ramond sector. Nevertheless, some introductory steps are common to both cases.

The kinetic operator $L_0^{\mathrm{tot}} = \mathcal{L}_0 + L_0 = \{b_0, D\}$ commutes with D and can be diagonalized in $\mathcal{C}(\mathcal{F}_\varepsilon)$. It is easy to see that any closed element outside $\ker L_0^{\mathrm{tot}}$ is exact. Therefore, the cohomology classes are determined by the co-chains of an on-mass-shell subcomplex:

$$\mathcal{C}^{(0)}(\mathcal{F}_\varepsilon) = \bigoplus_{N \in \mathbb{N}_\varepsilon} \int_{S_N} \mathrm{d}\mu^N(p) \mathcal{C}^N(\mathcal{F}_\varepsilon)(p). \tag{28}$$

The elements of $\mathcal{C}^N(\mathcal{F}_\varepsilon)(p)$ are supported on the mass shells S_N:

$$p^2 = -m_N^2 \qquad m_N^2 = 2\alpha\left(N + \frac{1}{2}\varrho^2 - \varepsilon\frac{d-1}{16}\right) \quad N = N^{\mathrm{str}} + N^{\mathrm{gh}} \tag{29}$$

where N^{str} and N^{gh} denote the eigenvalues of string and ghost level operator respectively. The direct integrals in (28) are taken with respect to the Lorentz-invariant measures on S_N. It is worth noting that all excited states ($N > 0$) from $\mathcal{C}^{(0)}(\mathcal{F}_\varepsilon)$ are generically massive. The only exception is $\varrho = 0$ and $d = 9$ in the NS sector, where the first excited level is massless.

The on-mass-shell complex is further reduced to the relative complex by eliminating the ghost zero modes (they do not contribute to the mass spectrum).

The space of a relative complex and the structure of the relative differential in the case of the NS sector are analogous to that of bosonic string theory [1]:

$$\mathcal{C}_{\mathrm{rel}}(\mathcal{F}_1)(p) := \mathcal{C}^{(0)}(\mathcal{F}_1)(p) \cap \ker b_0 \qquad D_{\mathrm{rel}} = D - L_0^{\mathrm{tot}} c_0 - M\, b_0 \qquad M = \{D, c_0\}. \tag{30}$$

Since $\mathcal{C}_{\mathrm{rel}}(\mathcal{F}_1)(p)$ does not contain the ghost zero modes it is convenient to shift the ghost number of the vacuum by $+\frac{1}{2}$ to obtain the relative grading by integers.

In the case of the Ramond complex one introduces a sequence of the following subspaces of $\mathcal{C}^{(0)}(\mathcal{F}_\varepsilon)(p)$:

$$\begin{aligned} &\mathcal{K}(\mathcal{F}_0)(p) := \mathcal{C}^{(0)}(\mathcal{F}_0)(p) \cap \ker b_0 \supset \mathcal{K}^{(0)}(\mathcal{F}_0)(p) := \mathcal{K}(\mathcal{F}_0)(p) \cap \ker \beta_0 \\ &\mathcal{C}_{\mathrm{rel}}(\mathcal{F}_0)(p) := \mathcal{K}^{(0)}(\mathcal{F}_0)(p) \cap \ker G_0^{\mathrm{tot}} \qquad G_0^{\mathrm{tot}} = [\beta_0, D]. \end{aligned} \tag{31}$$

The relative differential is given by

$$D_{\mathrm{rel}} = D - L_0^{\mathrm{tot}}\, c_0 - F\, \gamma_0 - (M - \gamma_0^2)\, b_0 - N\, \beta_0 \tag{32}$$

where $F = G_0^{\mathrm{tot}} + 2b_0\gamma_0$ is the restriction of Dirac–Ramond operator to the space $\mathcal{K}^{(0)}(\mathcal{F}_0)(p)$ while $M = \{D, c_0\} + \gamma_0^2$ and $N = [D, \gamma_0]$. All of these operators are free of the ghost zero modes.

An analogue of (30) in the Ramond sector—the space $\mathcal{K}(\mathcal{F}_0)(p)$—equipped with the differential $D_0 = D - c_0\, L_0^{\mathrm{tot}} - (M - \gamma_0^2) b_0$ is also a subcomplex of $\mathcal{C}^{(0)}(\mathcal{F}_0)(p)$. It is important to note, however, that in contrast to $\mathcal{C}_{\mathrm{rel}}(\mathcal{F}_1)(p)$ this space is still infinite dimensional. The infinite degeneracy is generated by arbitrary polynomials in bosonic ghost zero modes γ_0 and β_0, acting on the states which stem from $\omega_0(p)$ and $\omega_1(p)$ respectively. Fixing the level is not enough to obtain a finite-dimensional complex in the Ramond case. The property that makes the classical statement on Poincaré duality working in the case of $\mathcal{K}(\mathcal{F}_0)(p)$ is finite dimensionality at fixed level and at fixed ghost number.

This infinite degeneracy is removed in $\mathcal{K}^{(0)}(\mathcal{F}_0)(p)$. This space contains the states excited from $\omega_0(p)$ (the spinor factor is suppressed) by string and ghost modes with positive weights.

The elements of relative complex $\mathcal{C}_{\text{rel}}(\mathcal{F}_1)(p)$ of NS strings admit a simple description. They are generated by the polynomials of level N (with N given in terms of p by the mass shell condition (29)) in the original string excitation operators (1), (2) and ghost modes (15) out of the vacuum state $\omega(p)$.

Note that the scalar product of the total BRST complex is zero when restricted to the relative subspace. For this reason a non-degenerate scalar product on this complex is defined by

$$(\Psi, \Psi')_{\text{rel}} = (\Psi, (-1)^{F_1} c_0\, \Psi') \qquad \Psi, \Psi' \in \mathcal{C}_{\text{rel}}(\mathcal{F}_1)(p) \tag{33}$$

and one has $D^*_{\text{rel}} = D_{\text{rel}}$.

The space of relative complex in fact admits a richer, bigraded structure and decomposes into a direct sum of bihomogeneous components:

$$\mathcal{C}_{\text{rel}}(\mathcal{F}_1)(p) = \bigoplus_{\kappa} \mathcal{C}^{\kappa}_{\text{rel}}(\mathcal{F}_1)(p) \qquad \mathcal{C}^{\kappa}_{\text{rel}}(\mathcal{F}_1)(p) = \bigoplus_{a-b=\kappa} \mathcal{C}^{a}_{b}(\mathcal{F}_1)(p) \tag{34}$$

where κ is the relative ghost number and a, b denote the ghost (c, γ) and respectively anti-ghost (b, β) degree. The relative differential splits accordingly in the spirit of complex geometry [24]:

$$\begin{aligned} &D_{\text{rel}} = \overline{\mathcal{D}} + \mathcal{D} \qquad \overline{\mathcal{D}}^2 = 0 \qquad \mathcal{D}^2 = 0 \qquad \overline{\mathcal{D}}\,\mathcal{D} + \mathcal{D}\,\overline{\mathcal{D}} = 0 \\ &\overline{\mathcal{D}} : \mathcal{C}^{a}_{b}(\mathcal{F}_1)(p) \to \mathcal{C}^{a+1}_{b}(\mathcal{F}_1)(p) \qquad \mathcal{D} : \mathcal{C}^{a}_{b}(\mathcal{F}_1)(p) \to \mathcal{C}^{a}_{b-1}(\mathcal{F}_1)(p). \end{aligned} \tag{35}$$

The structures of $\overline{\mathcal{D}}$ and $\mathcal{D}$ are related to the decomposition of superconformal algebra (12) into subspaces of negative and positive roots: $S = S_- \oplus \mathbb{C}L_0 \oplus S_+$. The expression for $\overline{\mathcal{D}}$ in terms of elementary modes can be obtained from (11), (20) and (21), (25):

$$\overline{\mathcal{D}} = \sum_{m>0} (L_m + l_m) c_{-m} + \sum_{r>0} (G_r + g_r)\, \gamma_{-r} + \overline{\partial}. \tag{36}$$

The operator $\overline{\partial}$ is the canonical differential of S_- subalgebra:

$$\begin{aligned} \overline{\partial} = \frac{1}{2} \sum_{m>0} c_{-m} &\left(\sum_{k>0} (k-m) c_{-k}\, b_{m+k} + \sum_{r>0} \left(r - \frac{m}{2} \right) \gamma_{-s}\, \beta_{m+s} \right) \\ &+ \frac{1}{2} \sum_{s>0} \gamma_{-s} \left(\sum_{k>0} \left(s - \frac{k}{2} \right) c_{-k}\, \beta_{k+s} - 2 \sum_{r>0} \gamma_{-r}\, b_{r+s} \right). \end{aligned} \tag{37}$$

The operators L_m, G_r are the negative level string constraints (11) while their partners in (36):

$$\begin{aligned} l_m &= -\sum_{k>m} (m-k) c_{-k}\, b_{m-k} - \sum_{s>m} \left(s + \frac{m}{2} \right) \gamma_{-s}\, \beta_{m-s} \\ g_r &= \sum_{k>r} \left(r + \frac{k}{2} \right) c_k\, \beta_{s-k} - 2 \sum_{s>r} \gamma_{-s}\, b_{r-s} \end{aligned} \tag{38}$$

implement the coadjoint and adjoint actions of S_- on the ghosts and respectively anti-ghosts of S_+. The differentials are mutually adjoint with respect to the relative scalar product (33): $\mathcal{D}^* = \overline{\mathcal{D}}$.

An analogous construction in the Ramond sector is a little more complicated, as the space of the relative complex does not admit a direct description in terms of the original string and ghost modes. It is convenient, in this case, to introduce an explicit light-cone parametrization of the mass shells and to regard the condition $L_0^{\rm tot} = 0$ as an evolution equation.

Choose an arbitrary light-cone frame $\{e_\pm, e_i\}_{i=1}^{d-2}$. (This consists of two light-like vectors $e_\pm$; $e_\pm^2 = 0$, $e_+ \cdot e_- = -1$ and the Euclidean basis $\{e_i\}_1^{d-2}$ of $(d-2)$-dimensional transverse subspace $e_\pm \cdot e_i = 0$.) Any mass shell (29) can be (at least locally in the massless and tachyonic case) parametrized by the non-vanishing light-cone component $p^+ = e_+ \cdot p \neq 0$ and the transverse part $\overline{p}$ of the momentum. Choose $x^+ = e_+ \cdot x$ as an evolution parameter and put $P^- = {\rm i}\partial/\partial x^+$.

The space $\mathcal{K}^{(0)}(\mathcal{F}_0)(p)$ is thus generated out of evolving spinor vacuum states by time-dependent elementary modes with the evolution determined by their weights:

$$u(p^+, \overline{p}, x^+) = {\rm e}^{{\rm i}x^+ \frac{1}{2p^+}(\overline{p}^2 + \alpha\varrho^2)}\, u(p^+, \overline{p}) \qquad \mathcal{O}_m(x^+) = {\rm e}^{{\rm i}mx^+ \frac{\alpha}{p^+}}\, \mathcal{O}_m \tag{39}$$

where $\mathcal{O}$ denotes any of (1), (2) or (15).

The operator $G_0^{\rm tot}$ is nilpotent on $\mathcal{K}^{(0)}(\mathcal{F}_0)(p)$ and has trivial cohomology—there exists a 'contracting homotopy' Θ such that $\{G_0^{\rm tot}, \Theta\} = 1$. The 'contracting homotopy' can be most conveniently chosen by $\Theta = \sqrt{\alpha}\, d_0^+/p^+$, where d_0^+ denotes the light-cone component of the fermionic zero mode. Consequently,

$$\mathcal{C}_{\rm rel}(\mathcal{F}_0)(p) = G_0^{\rm tot}\, \mathcal{K}^{(0)}(\mathcal{F}_0)(p) \qquad \mathcal{K}^{(0)}(\mathcal{F}_0)(p) = \mathcal{C}_{\rm rel}(\mathcal{F}_0)(p) \,\oplus\, \Theta\, \mathcal{C}_{\rm rel}(\mathcal{F}_0)(p). \tag{40}$$

Following [15] one introduces the shifted modes:

$$\hat{\mathcal{O}}_m := [G_0^{\rm tot}, \Theta\, \mathcal{O}_m]_\pm = \mathcal{O}_m - \Theta[G_0^{\rm tot}, \mathcal{O}_m]_\mp. \tag{41}$$

The operation $\mathcal{O}_m \mapsto \hat{\mathcal{O}}_m$ induces an automorphism [27] of the associative operator algebra generated by all non-zero elementary modes: $\widehat{\mathcal{O}\mathcal{O}'} = \hat{\mathcal{O}}\hat{\mathcal{O}}'$. In particular, the map (41) preserves the canonical commutation relations.

The time-dependent partners $\hat{\mathcal{O}}_m(x^+)$ of (41) commute with $G_0^{\rm tot}$ and generate the relative states when acting upon the vacuum vectors satisfying the Dirac equation. Let $V(p)$ denote the space (see (A.4)) of Dirac vacuum states $G_0^{\rm tot} V(p) = 0$. It is not difficult to check that

$$\mathcal{C}_{\rm rel}(\mathcal{F}_0)(p) = \{\Psi;\, \Psi = W(\hat{\mathcal{O}}(x^+))v(p),\, v(p) \in V(p)\} \tag{42}$$

where $W(\cdot)$ is any polynomial in time-dependent, shifted creation operators. The right-pointed inclusion is obvious. From the structure of $V(p)$ (which is explicitly given in (A.4)) and the form of $\hat{\mathcal{O}}(x^+)$ it clearly follows that $\dim W(\hat{\mathcal{O}})V(p) = \frac{1}{2}\dim\, \mathcal{K}^{(0)}(\mathcal{F}_0)(p)$. This, together with (40), implies (42).

All subspaces defined in (31) may be equipped with non-degenerate scalar products. The subcomplex $\mathcal{K}(\mathcal{F}_0)(p)$ carries the pairing which is strictly analogous to that of (33):

$$(\Psi, \Psi')_{\mathcal{K}} = (\Psi, (-1)^{F_0}\, c_0\, \Psi) \qquad \Psi, \Psi' \in \mathcal{K}(\mathcal{F}_0)(p). \tag{43}$$

Since $\mathcal{K}^{(0)}(\mathcal{F}_0)(p)$ stems from ω_0 ghost vacuum, the form (43) is identically zero when restricted to this subspace.

In order to heal the scalar product one introduces an injective vacuum substitution operator $\chi : \mathcal{K}^{(0)}(\mathcal{F}_0)(p) \to \mathcal{K}(\mathcal{F}_0)(p)$, which exchanges the ghost vacuum factor $(\cdot) \otimes \omega_0$ of any state into $(\cdot) \otimes \omega_1$. The non-degenerate pairing on $\mathcal{K}^{(0)}(\mathcal{F}_0)(p)$ is then defined by

$$(\Psi, \Psi')_{\mathcal{K}^{(0)}} = (\chi(\Psi), \Psi')_{\mathcal{K}} \qquad \Psi, \Psi' \in \mathcal{K}^{(0)}(\mathcal{F}_0)(p). \tag{44}$$

The property $(G_0^{\rm tot}|_{\mathcal{K}^{(0)}})^* = -G_0^{\rm tot}|_{\mathcal{K}^{(0)}}$ with respect to (44) and the relation $(G_0^{\rm tot})^2 = L_0^{\rm tot}$ together with (40) imply that, again, the pairing $(,)_{\mathcal{K}^{(0)}}$ is zero when restricted to $\mathcal{C}_{\rm rel}(\mathcal{F}_0)(p)$. The non-degenerate, Hermitian scalar product on the Ramond relative complex is thus defined by the formulae

$$(\Psi, \Psi')_{\rm rel} = {\rm i}(\Psi, \Theta\, \Psi')_{\mathcal{K}^{(0)}} = {\rm i}(\chi(\Psi), (-1)^{F_0} c_0 \Theta \Psi') \qquad \Psi, \Psi' \in \mathcal{C}_{\rm rel}(\mathcal{F}_0)(p). \tag{45}$$

The above pairing has always a definite sign on the space $V(p)$ of Dirac vacuum vectors. The conventions used in appendix A fix it to be strictly positive. It is also important to note that the conjugation rules of the shifted modes with respect to (45) are changed and one has $\hat{\mathcal{O}}_m^* = \hat{\mathcal{O}}_{-m}$ independently of whether $\hat{\mathcal{O}}$ is of fermionic or bosonic type. These properties are crucial for the no-ghost theorem and, consequently, also for quantum mechanical interpretation of the relative cohomology states.

From the automorphism property of $\mathcal{O} \mapsto \hat{\mathcal{O}}$, it follows that any operator which commutes with G_0^{tot} can be rewritten in terms of shifted modes just by replacing the original ones by those of (41) in its expression. This applies in particular to the relative differential D_{rel} of the Ramond complex (32).

Introducing the bigrading of the type of (34) on the space $\mathcal{C}_{\mathrm{rel}}(\mathcal{F}_0)(p)$, one may split the relative differential (35): $D_{\mathrm{rel}} = \overline{\mathcal{D}} + \mathcal{D}$ with the expression for $\overline{\mathcal{D}}$ given by (36) and the formulae (37), (38) with all modes replaced by the shifted ones and obvious changes in the mode labels. Despite the fact that $\{D_{\mathrm{rel}}, \Theta\} \neq 0$ one has $(D_{\mathrm{rel}})^* = D_{\mathrm{rel}}$ and $(\overline{\mathcal{D}})^* = \mathcal{D}$. This property easily follows [17] from (40) and from the fact that $(\Psi, \Psi')_{\mathrm{rel}} \equiv (\Psi, (G_0^{\mathrm{tot}})^{-1}\Psi')_{\mathcal{K}^{(0)}}$ with $(G_0^{\mathrm{tot}})^{-1}\Psi'$ being any G_0^{tot}-primary of Ψ'.

Note, finally, that the local light-cone parametrization of the on-mass-shell complex by evolving modes (39) could be used in the case of the NS sector as well.

2. BRST cohomologies

This section is devoted to the calculation of the cohomology spaces introduced in the previous section. The vanishing of bigraded cohomologies will be shown first. Then the relative spaces will be identified and finally the absolute cohomology will be reconstructed.

2.1. Relative cohomology

As stated in the previous section, one may introduce the bigraded cohomology spaces in both sectors:

$$\overline{\mathcal{H}}_b^a(\mathcal{F}_\varepsilon)(p) = \frac{\overline{\mathcal{Z}}_b^a(\mathcal{F}_\varepsilon)(p)}{\overline{\mathcal{B}}_b^a(\mathcal{F}_\varepsilon)(p)} \qquad \mathcal{H}_b^a(\mathcal{F}_\varepsilon)(p) = \frac{\mathcal{Z}_b^a(\mathcal{F}_\varepsilon)(p)}{\mathcal{B}_b^a(\mathcal{F}_\varepsilon)(p)} \tag{46}$$

where $\overline{\mathcal{H}}$ and $\mathcal{H}$ denote the cohomologies of $\overline{\mathcal{D}}$ and $\mathcal{D}$ respectively.

The important and general property of the spaces (46) is that they are two-by-two isomorphic. More precisely, we have the following lemma.

Lemma 2.1 (Poincaré–Serre duality).

$$(\overline{\mathcal{H}}_b^a(\mathcal{F}_\varepsilon)(p))^* = \mathcal{H}_a^b(\mathcal{F}_\varepsilon)(p) \tag{47}$$

with the duality $(\cdot)^*$ *in the sense of relative pairing (33) or (45).*

Proof. A proof of the lemma follows from the fact that the scalar products on the relative complexes are non-degenerate and that $(\overline{\mathcal{D}})^* = \mathcal{D}$. This last property guarantees that $(\,,\,)_{\mathrm{rel}}$ induces a well defined (class representative independent) pairing of cohomology spaces. The finite dimensionality of $\mathcal{C}_b^a(\mathcal{F}_\varepsilon)(p)$ at fixed on-mass-shell momentum allows one to reduce the proof to simple algebraic statements. Some details of the reasoning were presented in [1]. □

In order to prove the vanishing theorems for (46) it is convenient and effective to use the technology of spectral sequences [23, 28] in a suitably simplified form [26]. Introduce a new gradation in the spaces of the complexes by assigning the filtration degrees to the elementary modes:

$$\begin{aligned}&\deg(a_m^+) = \deg(d_r^+) = \deg(c_m) = \deg(\gamma_r) = 1\\ &\deg(a_m^-) = \deg(d_r^-) = \deg(b_m) = \deg(\beta_r) = -1\\ &\deg(a_m^i) = \deg(d_r^{\mathrm{i}}) = \deg(u_m) = \deg(t_r) = 0 \qquad m, r \neq 0 \quad i = 1, \ldots, d-2\end{aligned} \tag{48}$$

where $(\cdot)^{\pm}$ denote the light-cone components of the vector modes. The same definition applies to the relative modes (41) of the Ramond sector. Note, however, that $\hat{d}_0^+ = 0$.

For simplicity, the common notation for the Neveu–Schwarz and Ramond relative modes will be assumed throughout this section.

The spaces $\mathcal{C}_b(\mathcal{F}_\varepsilon)(p)$ decompose into filtration degree homogeneous components $\mathcal{C}_b(\mathcal{F}_\varepsilon)(p) = \oplus_f \mathcal{C}_{b;f}(\mathcal{F}_\varepsilon)(p)$. Note that, for a momentum on the mass-shell S_N (29), the filtration degree is bounded by $-N \leqslant f \leqslant N$. The differential $\overline{\mathcal{D}}$ splits accordingly:

$$\begin{aligned}&\overline{\mathcal{D}} = \overline{\mathcal{D}}_{(0)} + \overline{\mathcal{D}}_{(1)} + \overline{\mathcal{D}}_{(2)}\\ &\overline{\mathcal{D}}_{(0)} = -\frac{1}{\sqrt{\alpha}}\, p^+ \left(\sum_{m>0} a_m^- c_{-m} + \sum_{r>0} d_r^-\, \gamma_{-r} \right)\\ &\overline{\mathcal{D}}_{(i)} : \mathcal{C}^a_{b;f}(\mathcal{F}_\varepsilon)(p) \to \mathcal{C}^{a+1}_{b;f+i}(\mathcal{F}_\varepsilon)(p).\end{aligned} \tag{49}$$

The operators $\overline{\mathcal{D}}_{(1)}$ and $\overline{\mathcal{D}}_{(2)}$ of higher filtration degrees can be easily read off from (11) and (36) but their explicit form is not used here. (They are not needed because of the vanishing theorem for filtered complex below.) Out of (49) only the nilpotent component $\overline{\mathcal{D}}_{(0)}$ will be important.

One introduces the cohomology spaces $\overline{\mathcal{H}}^a_{b;f}(\mathcal{F}_\varepsilon)(p)$ of $\overline{\mathcal{D}}_{(0)}$ localized at fixed filtration degree. It is not difficult to show that these cohomologies are trivial.

The operator $R_0 = \sum_{m>0}(m\, c_{-m} b_m - a^+_{-m} a^-_m) + \sum_{r>0} r(\gamma_{-r}\beta_r - d^+_{-r} d^-_r)$, of filtration degree zero, counts the level of a^+, d^+ excitations and c, γ ghost excitations. From the identity $\{\overline{\mathcal{D}}_{(0)}, \mathcal{J}\} = \frac{p^+}{\sqrt{\alpha}} R_0$, where $\mathcal{J} = \sum_{m>0} a^+_{-m} b_m - \sum_{r>0} r\, d^+_{-r}\beta_r$ one easily concludes that all co-chains outside the kernel of R_0, in particular those with $a > 0$, are cohomologically trivial. Hence

$$\overline{\mathcal{H}}^a_{b;f}(\mathcal{F}_\varepsilon)(p) = 0 \qquad a > 0 \quad p \neq 0. \tag{50}$$

The simple statement above makes it possible, as in the case of bosonic string theory [1], to prove an analogue of the Dolbeaut–Grotendieck lemma of classical complex geometry [24] on the vanishing of bigraded cohomologies (46).

Lemma 2.2 (Dolbeaut–Grotendieck).

$$\overline{\mathcal{H}}^a_b(\mathcal{F}_\varepsilon)(p) = 0 = \mathcal{H}^b_a(\mathcal{F}_\varepsilon)(p) \qquad a > 0 \quad p \neq 0. \tag{51}$$

Proof. The result for the cohomologies of $\overline{\mathcal{D}}$ is obvious in the light of general theorems on cohomologies of filtered complexes [23, 28]. An elementary argument, based on the general ideas of reasoning [31] was presented in the context of bosonic theory in [1, 26] and is repeated here for the sake of completeness. Any cohomology class $[\Psi^a] \in \overline{\mathcal{H}}^a_b(\mathcal{F}_\varepsilon)(p)$ is represented by a co-cycle $\Psi^a = \sum_{f \geqslant m} \Psi^a_{*;\,f}$ with $\overline{\mathcal{D}}_{(0)}$-closed lowest filtration degree component $\Psi^a_{*;\,m}$. Because of (50), there exists $\overline{\mathcal{D}}_{(0)}$-primary φ^{a-1}_m of this element. The lowest filtration degree component of an equivalent co-cycle $\Psi'^a = \Psi^a - \overline{\mathcal{D}}\varphi^{a-1}_m$ is of order at most $m-1$. Using (50) one may, step by step, prove that $\Psi^a \sim \overline{\mathcal{D}}\Phi^{a-1}$ for some Φ^{a-1}. Hence $\overline{\mathcal{H}}^a_b(\mathcal{F}_\varepsilon)(p) = 0$; $a > 0$.

The right-hand side equality follows from the Poincaré–Serre duality of lemma 2.1. □

The result above implies directly the vanishing theorem for relative cohomology classes

$$H^\kappa_{\mathrm{rel}}(\mathcal{F}_\varepsilon)(p) = \frac{Z^\kappa_{\mathrm{rel}}(\mathcal{F}_\varepsilon)(p)}{B^\kappa_{\mathrm{rel}}(\mathcal{F}_\varepsilon)(p)} \tag{52}$$

and gives a convenient description of the non-vanishing classes in terms of bigraded co-cycles.

Theorem 2.1 (Vanishing theorem).

$$(1)\ H^{\kappa}_{\rm rel}(\mathcal{F}_\varepsilon)(p) = 0 \qquad \kappa \neq 0 \quad p \neq 0$$

$$(2)\ H^{0}_{\rm rel}(p) \sim \overline{\mathcal{Z}}^0_0(\mathcal{F}_\varepsilon)(p)/\mathcal{D}\overline{\mathcal{Z}}^0_1(\mathcal{F}_\varepsilon)(p).$$

Proof.

(1) The reasoning here is, in fact, a copy of that of lemma 2.2. Any co-cycle $\Psi^\kappa \in Z^\kappa_{\rm rel}(\mathcal{F}_\varepsilon)(p)$; $\kappa \geqslant 0$ can be decomposed into bihomogeneous components $\Psi^\kappa = \sum_{b=0}^{m} \Psi^{\kappa+b}_b$ with the highest one satisfying $\overline{\mathcal{D}}\Psi^{\kappa+m}_m = 0$. Because of (51) one has $\Psi^{\kappa+m}_m = \overline{\mathcal{D}}\Phi^{\kappa+m-1}_m$. The equivalent co-cycle $\Psi'^\kappa = \Psi^\kappa - D_{\rm rel}\Phi^{\kappa+m-1}_m$ does not contain the term of bidegree $(\kappa+m, m)$. Continuing the elimination procedure by induction one concludes that $\Psi^\kappa \sim 0$ unless $\kappa = 0$. The vanishing in the case of negative ghost number follows from the Poincaré duality for relative cohomologies.

(2) A similar argument as above leads to the statement that $Z^0_{\rm rel}(\mathcal{F}_\varepsilon)(p) \ni \Psi^0 \sim \Psi^0_0 \in \overline{\mathcal{Z}}^0_0(\mathcal{F}_\varepsilon)(p)$. Two co-cycles Ψ^0_0 and Ψ'^0_0 are equivalent in the sense of relative cohomology iff $\Psi^0_0 - \Psi'^0_0 = \mathcal{D}\Phi^0_1$ with $\overline{\mathcal{D}}\Phi^0_1 = 0$. □

It is worth noting that the second equality in the vanishing theorem allows one to establish an introductory relation between the relative cohomology classes and the space of physical states. The set $\overline{\mathcal{Z}}^0_0(\mathcal{F}_\varepsilon)(p)$ can be easily identified with the space (14) of the physical vectors of 'old covariant' formalism. The state $\Psi^0_0(p) = \varphi(p) \otimes \omega$; $\varphi(p) \in \mathcal{F}_\varepsilon(p)$ (with ω denoting the appropriate ghost vacuum) is $D_{\rm rel}$ closed if and only if $\sum_{m>0} L_m\varphi(p) \otimes c_{-m}\omega + \sum_{r>0} G_r\varphi(p) \otimes \gamma_{-r}\omega = 0$. Hence

$$\overline{\mathcal{Z}}^0_0(\mathcal{F}_\varepsilon)(p) = \mathcal{F}^{\rm phys}_\varepsilon(p) \otimes \omega. \tag{53}$$

The space $\mathcal{D}\overline{\mathcal{Z}}^0_1(p)$ of 'pure gauge' elements in (53) will be later identified with the set of null vectors in (14).

The vanishing theorem gives one the possibility to use a simple method [16, 30] to determine the dimensions of the relative cohomology spaces. They can be computed with the help of the Euler–Poincaré principle provided the complexes under consideration are of finite dimension. This is the case for the relative complex $\mathcal{C}_{\rm rel}(\mathcal{F}_\varepsilon)(p)$ at fixed on-mass-shell momentum and with its Euler–Poincaré characteristic ${\rm ch}_\varepsilon(p)$ satisfying the sequence of identities:

$${\rm ch}_\varepsilon(p) := \sum_\kappa (-1)^\kappa \dim \mathcal{C}^\kappa_{\rm rel}(\mathcal{F}_\varepsilon)(p) = \sum_\kappa (-1)^\kappa \dim H^\kappa_{\rm rel}(\mathcal{F}_\varepsilon)(p) = \dim H^0_{\rm rel}(\mathcal{F}_\varepsilon)(p). \tag{54}$$

The second equality above is the expression for the Euler–Poincaré principle [23,31], while the third one is implied by the vanishing of higher cohomologies. The dimensions of $H^0_{\rm rel}(\mathcal{F}_\varepsilon)(p)$ can be thus described in terms of a generating series for the Euler–Poincaré characteristic of the relative complex:

$${\rm ch}_\varepsilon(q) = q^{\frac{1}{2\alpha}p^2 + \frac{1}{2}\varrho^2 - \varepsilon\frac{d-1}{16}}\, E_\varepsilon(q) \qquad {\rm ch}_\varepsilon(p) = {\rm ch}_\varepsilon(q)|_0 \tag{55}$$

where $|_0$ denotes the constant term. The series $E_\varepsilon(q)$ can be constructed as a product of the generating series (13) for dimensions of $\mathcal{F}_\varepsilon(p)$ and the generating function for the alternating sums of ghost contributions appearing on the left-hand side of (54):

$$E^{\rm gh}_\varepsilon(q) = E^2_+(q)\, E^2_{\varepsilon-}(q) \qquad E_+(q) = \prod_{n>0}(1 - q^n) \quad E_{\varepsilon-}(q) = \prod_{n>0}(1 + q^{n-\frac{\varepsilon}{2}})^{-1}. \tag{56}$$

The functions above can be, in turn, easily obtained from the generating series $P_{\varepsilon\mp}(q, t)$ in two variables (t, q) corresponding to a family of elementary modes. The powers of

t keep track of the ghost number of the states while the powers of q count the level: $P_{\varepsilon\mp}(q,t) = \prod_{n>0}(1 \mp t^{\mathrm{gh}} q^{n-\frac{\varepsilon}{2}})^{\mp 1}$, where gh $= \pm 1, 0$ is the ghost number of the family of modes. The alternating sums of (54) are obtained by putting $t = -1$.

The final formula for dimension reads

$$\dim H^0_{\mathrm{rel}}(\mathcal{F}_\varepsilon)(p) = q^{\frac{1}{2\alpha}p^2 + \frac{1}{2}\varrho^2 - \varepsilon\frac{d-1}{16}} 2^{(1-\varepsilon)([\frac{d+2}{2}]-1)} \prod_{n>0}(1-q^n)^{-d+1} \prod_{n>0}(1+q^{n-\frac{\varepsilon}{2}})^{d-1}|_0. \qquad (57)$$

Note that the coefficient describing the multiplicity of Ramond ground states is two times smaller than that of (13), which is a reflection of the fact that vacuum states of relative complex do satisfy the Dirac equation. Formula (57) will be useful for the proof of the no-ghost theorem in the next section.

2.2. Absolute cohomology

The reconstruction of the absolute cohomology spaces out of those of the relative complex is much more complicated in the Ramond case. It will be performed here in two steps. The Neveu–Schwarz case can then be obtained by following and simplifying the considerations for Ramond theory.

In order to recover the absolute cohomologies of the Ramond sector it is convenient to consider an intermediate complex and the corresponding cohomology spaces. The space of its co-chains $\mathcal{K}(\mathcal{F}_0)(p)$ was already introduced (31) in the course of construction of the relative complex:

$$\mathcal{K}(\mathcal{F}_0)(p) = \mathcal{K}_0(\mathcal{F}_0)(p) \oplus \mathcal{K}_1(\mathcal{F}_0)(p) \qquad D_0 = D_{\mathrm{rel}} + F\gamma_0 + N\beta_0. \qquad (58)$$

The subspaces $\mathcal{K}_\lambda(\mathcal{F}_0)(p);\ \lambda = 0, 1$ stem from respective ghost vacuum vectors of (17). The differential in $\mathcal{K}(\mathcal{F}_0)(p)$ is simply the restriction of the absolute one to this subspace. It is convenient to assume that the above complex is graded by the ghost number operator $\mathrm{gh}_{\mathcal{K}} = \mathrm{gh}_{\mathrm{rel}} + \frac{1}{2}(\beta_0\gamma_0 + \gamma_0\beta_0) = \mathrm{gh}_0 - \frac{1}{2}(c_0b_0 - b_0c_0)$ such that $\mathrm{gh}_{\mathcal{K}}\,\omega_\lambda = (\frac{1}{2} - \lambda)\,\omega_\lambda$. Note also that $\mathcal{K}^{(0)}(\mathcal{F}_0)(p) \subset \mathcal{K}_0(\mathcal{F}_0)(p)$. It is worth recalling that the spaces $\mathcal{K}_\lambda(\mathcal{F}_0)(p)$ are infinite-dimensional, but $\mathcal{K}^\kappa_\lambda(\mathcal{F}_0)(p)$ of fixed ghost number are of finite dimension.

The following technical lemma [27] is of primary importance.

Lemma 2.3. *Let* $\Psi^\kappa \in \mathcal{K}_0(\mathcal{F}_0)(p)$.

If $D_0\Psi^\kappa \in \mathcal{K}^{(0)}(\mathcal{F}_0)(p)$ *then* $\Psi^\kappa = D_0 f^{\kappa-1} + f^\kappa_{\mathrm{rel}}$ *for some* $f^{\kappa-1} \in \mathcal{K}_0(\mathcal{F}_0)(p)$ *and* $f^\kappa_{\mathrm{rel}} \in \mathcal{C}_{\mathrm{rel}}(\mathcal{F}_0)(p)$.

Proof. Assume the expansion $\Psi^\kappa = \sum_{n=1}^m \gamma_0^n \Phi^\kappa_n$; $\mathrm{gh}_{\mathcal{K}}(\Phi^\kappa_n) = k - n$. The condition $D_0\Psi^\kappa \in \mathcal{K}^{(0)}(\mathcal{F}_0)(p)$ implies a chain of equations:

$$\begin{aligned}
&F\Phi^\kappa_m = 0\\
&D_{\mathrm{rel}}\Phi^\kappa_m + \Phi^\kappa_{m-1} = 0\\
&\vdots\\
&D_{\mathrm{rel}}\Phi^\kappa_1 + F\Phi^\kappa_0 + 2N\Phi^\kappa_2 = 0
\end{aligned}$$

obtained by demanding the coefficients of all positive powers of γ_0 in $D_0\Psi^\kappa$ to vanish. The first equation is solved by $\Phi^\kappa_m = Ff^\kappa_m$ for some (by no means unique) $f^\kappa_m \in \mathcal{K}^{(0)}(\mathcal{F}_0)(p)$. Substituting this solution into the next equation one obtains $\Phi^\kappa_{m-1} = D_{\mathrm{rel}}f^\kappa_m + Ff^\kappa_{m-1}$ with some $f^\kappa_{m-1} \in \mathcal{K}^{(0)}(\mathcal{F}_0)(p)$. Using the crucial property of F being exact, one may continue by induction to obtain

$$\Phi^\kappa_l = D_{\mathrm{rel}}f^\kappa_{l+1} + (l+1)Nf^\kappa_{l+2} + Ff^\kappa_l \qquad 0 \leqslant l \leqslant m \quad f^\kappa_l = 0 \quad l > m \quad l < 0.$$

It is then straightforward to check that

$$\Psi^\kappa = D_0 f^{\kappa-1} + F f_0^\kappa \qquad \text{where} \quad f^{\kappa-1} = \sum_{n=0}^{m-1} \gamma_0^n \, f_{n+1}^\kappa . \qquad \Box$$

Let $Z_0^\kappa(\mathcal{F}_0)(p)$ denote the space of D_0 co-cycles from $\mathcal{K}_0(\mathcal{F}_0)(p)$. From the lemma above one may immediately conclude that any D_0 co-cycle is equivalent to a relative one.

Corollary 2.1. *Let* $\Psi^\kappa \in Z_0^\kappa(\mathcal{F}_0)(p)$.
Then $\Psi^\kappa = D_0 f^{\kappa-1} + \Psi^\kappa_{\rm rel}$, *where* $\Psi^\kappa_{\rm rel} \in Z^\kappa_{\rm rel}(\mathcal{F}_0)(p)$.

Proof. Use lemma and impose $D_0\Psi^\kappa = 0$. Then $0 = D_0(f^\kappa_{\rm rel}) = D_{\rm rel}(f^\kappa_{\rm rel})$. □

The next statement is a simple consequence of the one above. It will be shown that the natural inclusion map:

$$Z^\kappa_{\rm rel}(\mathcal{F}_0)(p) \ni \Psi^\kappa \to i_+(\Psi^\kappa) := \Psi^\kappa \in Z_0^{\kappa+\frac{1}{2}}(\mathcal{F}_0)(p) \tag{59}$$

induces an isomorphism of corresponding cohomology spaces.

Corollary 2.2. *The map*

$$H^0_{\rm rel}(F_0)(p) \ni [\Psi] \to i_+^*[\Psi] = [i_+(\Psi)]_{\mathcal{K}} \in H_0^{\frac{1}{2}}(\mathcal{F}_0)(p)$$

injectively covers the cohomology of $D_0|_{\mathcal{K}_0(\mathcal{F}_0)(p)}$.

Proof. It is clear that $i_+ D_{\rm rel} = D_0 \, i_+$. Hence i_+ transforms co-cycles into co-cycles. From corollary 2.1 it follows that any D_0 closed element is D_0 equivalent to a relative co-cycle. Since the relative differential is a restriction of D_0 to the relative subcomplex one has $H_0^{\kappa+\frac{1}{2}}(\mathcal{F}_0)(p) = 0; \kappa \neq 0$.

For $\kappa = 0$ it is enough to demonstrate that the map is injective. According to corollary 2.1 the co-cycle $\Psi^{\frac{1}{2}}$ is always equivalent to the image of some relative one under i_+. If it is exact in the sense of D_0 cohomology then, because i_+ intertwines the differentials, the corresponding relative co-cycle is trivial too. □

The full cohomology of $\mathcal{K}(\mathcal{F}_0)(p)$ can be recovered by use of the Poincaré duality principle with respect to $(\,,\,)_{\mathcal{K}}$ of (43). The spaces $\mathcal{K}_0(\mathcal{F}_0)(p)$ and $\mathcal{K}_1(\mathcal{F}_0)(p)$ are paired in a non-degenerate way with respect to this form. The principle implies that the only non-trivial cohomology space of $\mathcal{K}_1(\mathcal{F}_0)(p)$ is $H_1^{-1/2}(\mathcal{F})(p)$—the Poincaré dual of $H_0^{1/2}(\mathcal{F})(p)$.

In order to obtain more explicit information on the content of this cohomology space one may define a mapping from the space $\overline{\mathcal{Z}}{}^0_0(\mathcal{F}_0)(p)$ of theorem 2.1 into the space $Z_1^{-1/2}(\mathcal{F}_0)(p)$ of D_0 closed co-chains, namely

$$\overline{\mathcal{Z}}{}^0_0(\mathcal{F}_0)(p) \ni \Psi_0^0 \to i_-(\Psi_0^0) := \Theta\, \chi(\Psi_0^0) + \beta_0 \, Q \, \chi(f_0^0) \in Z_1^{-\frac{1}{2}}(\mathcal{F}_0)(p). \tag{60}$$

The operator Θ in the above formula denotes the contracting homotopy of F from (40) and $Q = \{D_{\rm rel}, \Theta\}$. The element f_0^0 is an arbitrarily chosen F-primary of Ψ_0^0: $F f_0^0 = \Psi_0^0$ and χ denotes the vacuum substitution map. In order to check that (60) transforms co-cycles into closed elements one should take into account that $N\,\chi(\Psi_0^0) = 0$ for any co-chain of bidegree $(0, 0)$ and that $[D_{\rm rel}, Q] = [F, Q] = [N, Q] = 0$. The last three equations follow immediately from the definition of Q and graded Jacobi identities.

The analogous result to that of corollary 2.2 can be proved for the injection (60).

Corollary 2.3. *The map*

$$H^0_{\mathrm{rel}}(F_0)(p) \ni [\Psi^0_0] \to i^*_-[\Psi^0_0] = [i_-(\Psi^0_0)]_{\mathcal{K}} \in H_1^{-\frac{1}{2}}(\mathcal{F}_0)(p)$$

injectively covers the cohomology of $D_0|_{\mathcal{K}_1(\mathcal{F}_0)(p)}$.

Proof. Assume that Ψ^0_0 represents a non-trivial element of relative cohomology and $i_-(\Psi^0_0) = D_0 f^{-\frac{3}{2}}$ in $\mathcal{K}_1(\mathcal{F}_0)(p)$. Then (by abusing the notation a little)

$$0 = (i_-(\Psi^0_0), H_0^{\frac{1}{2}}(\mathcal{F}_0)(p))_{\mathcal{K}} = (\Theta\chi(\Psi^0_0), (-1)^{F_0}\, c_0\, i^*_+(H^0_{\mathrm{rel}}(F_0)(p)).$$

But this implies (45) $(\Psi^0_0, H^0_{\mathrm{rel}}(\mathcal{F}_0)(p))_{\mathrm{rel}} = 0$ and yields a contradiction. □

The considerations above can be summarized by stating that full cohomology of D_0 is given by the direct sum

$$H_{\mathcal{K}}(\mathcal{F}_0)(p) = H_{\mathcal{K}}^{-\frac{1}{2}}(\mathcal{F}_0)(p) \oplus H_{\mathcal{K}}^{\frac{1}{2}}(\mathcal{F}_0)(p) \tag{61}$$

where the spaces on the right-hand side are explicitly expressed (59), (60) as the isomorphic images of relative cohomology group.

The reconstruction of absolute BRST cohomologies out of those of (61) follows, in fact, an analogous line of reasoning as that which led from the relative space towards recovery of $H_{\mathcal{K}}(\mathcal{F}_0)(p)$.

As indicated in (31) the absolute on-mass-shell complex decomposes into a direct sum:

$$\mathcal{C}^{(0)}(\mathcal{F}_0)(p) = \mathcal{K}(\mathcal{F}_0)(p) \oplus c_0\,\mathcal{K}(\mathcal{F}_0)(p) \tag{62}$$

and there are two natural injections of the space $\mathcal{K}(\mathcal{F}_0)(p)$ into the absolute complex. The first one, denoted by j, is given by the canonical inclusion of this space as a first summand in (62). The second map is defined as the multiplication of every element by the zero-mode c_0. The injection j intertwines the differentials $Dj = jD_0$ and, in contrast to the multiplication map, transforms the D_0 co-cycles into absolute ones. The intertwining property of j guarantees that the map j^* induced on cohomologies is well defined.

In order to define the counterpart of the multiplication map with similar properties one should first note that if Ψ^κ is a D_0 co-cycle, then there always exists some primary $h^{\kappa+1}$ of $(M-\gamma_0^2)\Psi^\kappa$. If $\kappa \neq \pm\frac{1}{2}$, then due to (61) $\Psi^\kappa = D_0 f^{\kappa-1}$ and $h^{\kappa+1} = (M-\gamma_0^2) f^{\kappa-1}$. Otherwise, the element $(M-\gamma_0^2)\Psi^\kappa$ is of degree $\neq \pm\frac{1}{2}$ and is closed under D_0. Hence it is trivial.

Using the above simple fact one may define a properly modified multiplication map

$$Z^\kappa_{\mathcal{K}}(\mathcal{F}_0)(p) \ni \Psi^\kappa \;\to\; c(\Psi^\kappa) := c_0\Psi^\kappa - j(h(\Psi^\kappa)) \in Z^{\kappa+\frac{1}{2}}(\mathcal{F}_0)(p) \tag{63}$$

by choosing the appropriate primaries $h(\Psi^\kappa)$ for all D_0 co-cycles. This choice can always be made in such a way that $c(\Psi^\kappa + D_0 f^{\kappa-1}) = c(\Psi^\kappa) - D(c_0 f^{\kappa-1})$. Then c induces a well defined map c^* of cohomology spaces.

First it will be demonstrated that the absolute cohomologies can be non-trivial only at the absolute ghost number κ satisfying $-1 \leqslant \kappa \leqslant 1$. Assume that $\kappa > 1$ and let $\Psi^\kappa = j(\Phi^{\kappa+1/2}) + c_0 j(\Phi^{\kappa-1/2})$ be an absolute co-cycle. (An element of $\mathcal{K}$ has the ghost number shifted by $-\frac{1}{2}$ when looked upon as an absolute co-chain.) Then its components satisfy $D_0\Phi^{\kappa-1/2} = 0$ and $D_0\Phi^{\kappa+1/2} + (M-\gamma_0^2)\Phi^{\kappa-1/2} = 0$. Since D_0 cohomology vanishes for $\kappa - \frac{1}{2} > \frac{1}{2}$, one has $\Phi^{\kappa-1/2} = D_0 f^{\kappa-3/2}$ for some $f^{\kappa-3/2}$. The equivalent co-cycle $\Psi'^\kappa = \Psi^\kappa + D(c_0 f^{\kappa-3/2})$ does not contain the $c_0\,(\cdot)$ term and is the image of some D_0 co-cycle under j: $\Psi'^\kappa = j(\Phi'^{\kappa+1/2})$. Because of the intertwining property of j and (62), Ψ'^κ is trivial. An analogous reasoning gives the conclusion for $\kappa < -1$.

The reconstruction of absolute cohomologies at the end points $\kappa = \pm 1$ of the admissible range is almost immediate. The closure of $\Psi^{-1} = j(\Phi^{-1/2}) + c_0 j(\Phi^{-3/2})$ implies, as above, $\Psi^{-1} \sim j(\Phi'^{-1/2})$. Further identification of co-cycles is equivalent to that in D_0 cohomology. Hence $H^{-1}(\mathcal{F}_0)(p) = j^* H_{\mathcal{K}}^{-1/2}(\mathcal{F}_0)(p)$. An analogous result, $H^1(\mathcal{F}_0)(p) = c^* H_{\mathcal{K}}^{1/2}(\mathcal{F}_0)(p)$ for $\kappa = 1$, follows from Poincaré duality, but can also be easily obtained by direct reasoning. For Ψ^1 to be closed it is necessary that $D_0\Phi^{1/2} = 0$ and $D_0\Phi^{3/2} + (M - \gamma_0^2)\Phi^{1/2} = 0$. Note, however, that the second term is exact and its primary $h(\Phi^{1/2})$ can be chosen according to the convention applied in definition (63). The sum $\Phi^{3/2} + h(\Phi^{1/2})$ is a co-cycle of degree $\frac{3}{2}$ and must be trivial. Let $f^{1/2}$ be its primary. Then $\Psi^1 \sim \Psi'^1 = \Psi^1 - Dj(f^{1/2}) = c(\Phi^{1/2})$ and one obtains the desired result.

The situation is a bit more complicated at $\kappa = 0$. The absolute co-cycle of ghost number zero is of the form $\Psi^0 = j(\Phi^{1/2}) + c_0 j(\Phi^{-1/2})$, and in particular, $D_0\Phi^{-1/2} = 0$. The element $\Phi^{-1/2}$ has two independent components $\Phi_\lambda^{-1/2}$; $\lambda = 0, 1$ built up over different vacua. From (62) it is clear that $\Phi_0^{-1/2} = D_0 f_0^{-3/2}$. An equivalent co-cycle $\Psi'^0 = \Psi^0 - Dc_0 j(f_0^{-3/2})$ is free of the component $c_0 j(\Phi_0^{-1/2})$ and the remaining ones should satisfy $D_0\Phi_0^{1/2} = 0$, $D_0\Phi_1^{-1/2} = 0$ and $D_0\Phi_1^{1/2} + (M - \gamma_0^2)\Phi_1^{-1/2} = 0$. A similar argument as that used in the case of $\kappa = 1$ leads to the conclusion that $\Phi_1^{1/2} + h(\Phi_1^{-1/2}) = D_0 f^{-1/2}$ (63). Hence an equivalent co-cycle $\Psi''^0 = \Psi'^0 - Dj(f^{-1/2})$ is of the form $\Psi''^0 = j(\Phi_0^{1/2}) + c(\Phi_1^{-1/2})$ and the remaining 'gauge' freedom is reduced to D_0 cohomological equivalence in both components independently. (The co-cycle $\Phi_1^{-1/2}$ must be in the kernel of $M - \gamma_0^2$, which can always be satisfied.) The absolute cohomology space at ghost number zero is thus a direct sum of two spaces $H^0(\mathcal{F}_0)(p) = H_0^0(\mathcal{F}_0)(p) \oplus H_1^0(\mathcal{F}_0)(p)$ which stem from different ghost vacuum vectors and are given as images of the appropriate spaces (62) under j^* and c^* respectively.

The considerations can be summarized in the form of the following theorem, which exhibits the explicit relation between the relative cohomology space and those of the absolute complex at non-zero momentum $p \neq 0$.

Theorem 2.2 (Absolute cohomology—Ramond sector). *The mappings $i_\pm^*$ and j^*, c^* of the following diagram:*

$$\begin{array}{ccccccccc} H_1^0(\mathcal{F}_0)(p) & & & & & & & & H_0^0(\mathcal{F}_0)(p) \\ & \overset{c^*}{\nwarrow} & & & & & & \overset{j^*}{\nearrow} & \\ & & H_{\mathcal{K}}^{-\frac{1}{2}}(\mathcal{F}_0)(p) & \overset{i_-^*}{\longleftarrow} & H_{\mathrm{rel}}^0(\mathcal{F}_0)(p) & \overset{i_+^*}{\longrightarrow} & H_{\mathcal{K}}^{\frac{1}{2}}(\mathcal{F}_0)(p) & & \\ & \overset{j^*}{\swarrow} & & & & & & \overset{c^*}{\searrow} & \\ H^{-1}(\mathcal{F}_0)(p) & & & & & & & & H^1(\mathcal{F}_0)(p) \end{array}$$

cover injectively the absolute cohomology space of the Ramond sector for $p \neq 0$.

Since all the mappings of the above diagram are isomorphisms, it describes, in fact, the replication of relative classes to form the space of the absolute cohomology.

An analogous result for the NS complex is much easier to obtain. It is enough to define the inclusion map j and the multiplication map c (63) from the space of relative co-cycles into the space of absolute complex. Then arguments similar to those preceding theorem 2.2 prove the following theorem.

Theorem 2.3 (Absolute cohomology—Neveu–Schwarz sector). *The mappings j^* and c^* of the following diagram:*

$$H^{-\frac{1}{2}}(\mathcal{F}_1)(p) \overset{j^*}{\longleftarrow} H_{\mathrm{rel}}^0(\mathcal{F}_1)(p) \overset{c^*}{\longrightarrow} H^{\frac{1}{2}}(\mathcal{F}_1)(p)$$

cover injectively the absolute cohomology space of the Neveu–Schwarz sector for $p \neq 0$.

In order to complete the analysis of BRST cohomologies one should determine the non-zero classes at $p = 0$. The mass-shell condition (29) restricts the necessary considerations to the cases when $m_N^2 = 2\alpha(N + \frac{1}{2}\varrho^2 - \varepsilon\frac{d-1}{16}) = 0$ and eliminates almost all states from the game. Note, first, that for $\varrho \neq 0$, the cohomology is zero. The only possibilities are left at level $N = \frac{1}{2}$ in $d = 9$ for the NS sector and at level $N = 0$ in the Ramond case, independently of d.

The complex $\mathcal{C}^{(0)}(\mathcal{F}_1)(0)$ is of finite dimension and is spanned by the following subspaces:

$$\mathcal{C}_{\mathrm{NS}}^{-\frac{1}{2}} = V \oplus \mathbb{C}\, c_0\beta_{-\frac{1}{2}}\omega(0) \qquad \mathcal{C}_{\mathrm{NS}}^{\frac{1}{2}} = c_0 V \oplus \mathbb{C}\,\gamma_{-\frac{1}{2}}\omega(0)$$

$$\mathcal{C}_{\mathrm{NS}}^{-\frac{3}{2}} = \mathbb{C}\,\beta_{-\frac{1}{2}}\omega(0) \qquad \mathcal{C}_{\mathrm{NS}}^{\frac{3}{2}} = \mathbb{C}\, c_0\gamma_{-\frac{1}{2}}\omega(0) \qquad V = \mathbb{C}\{d^{\mu}_{-\frac{1}{2}}\omega(0), t_{-\frac{1}{2}}\omega(0)\}.$$

The complex $\mathcal{C}^{(0)}(\mathcal{F}_0)(0)$ of the Ramond sector is infinite-dimensional and is given as a direct sum of the following subspaces:

$$S_0^m = \mathbb{C}\gamma_0^m\, S_0 \qquad S_1^m = \mathbb{C}\beta_0^m\, S_1 \qquad \text{and} \qquad c_0 S_0^m \qquad c_0 S_1^m \qquad m \geqslant 0$$

where $S_\lambda = S(d+1, 1) \otimes \omega_\lambda$ are the spinor modules with appropriate ghost vacuum roots. Note that the differential restricted to this subcomplex reduces to $D = -\gamma_0^2\, b_0$.

By direct calculation one may show the following theorem.

Theorem 2.4 (Absolute cohomology $p = 0$).

$$H^{-\frac{1}{2}}(\mathcal{F}_1)(0) \simeq V \qquad H^{\frac{1}{2}}(\mathcal{F}_1)(0) \simeq c_0 V \qquad H^{\pm\frac{3}{2}}(\mathcal{F}_1)(0) \simeq \mathcal{C}_{\mathrm{NS}}^{\pm\frac{3}{2}} \qquad d = 9.$$

$$H^{-2}(\mathcal{F}_0)(0) \simeq S_1^1 \qquad H^{-1}(\mathcal{F}_0)(0) \simeq c_0 S_1^1 \qquad H^{0}(\mathcal{F}_0)(0) \simeq S_0^0 \oplus c_0 S_1^0$$

$$H^{2}(\mathcal{F}_0)(0) \simeq c_0 S_0^1 \qquad H^{1}(\mathcal{F}_0)(0) \simeq S_0^1 \qquad 1 < d < 10.$$

The structure of the cohomology space at $p = 0$ is in the NS case similar to that of critical string theory [17] only in nine-dimensional spacetime. The cohomology of Ramond complex is, independently of dimension, described by the direct sums of appropriate spinor modules, as above.

3. The no-ghost theorem, cohomology representations and GSO projections

The scalar product (33) or (45) on the relative complex induces a pairing on the space of relative cohomologies via their representatives:

$$\langle [\Psi(p)]_{\mathrm{rel}}, [\Psi'(p)]_{\mathrm{rel}}\rangle = (\Psi(p), \Psi'(p))_{\mathrm{rel}} \qquad \Psi(p), \Psi'(p) \in Z^0_{\mathrm{rel}}(\mathcal{F}_\varepsilon)(p). \tag{64}$$

It is well defined, i.e. does not depend on the choice of representing co-cycles, due to the fact that $D^*_{\mathrm{rel}} = D_{\mathrm{rel}}$ with respect to $(,)_{\mathrm{rel}}$ independently of the sector. The Poincaré duality guarantees that it is non-degenerate.

The positivity of (64) can by proved by comparing [30] the Euler–Poincaré characteristic of the relative complex (54) with its signature. The signature of the space equipped with a non-degenerate Hermitian form is defined as the trace of its matrix in an orthonormal basis.

Hence the positivity of the scalar product (64) is equivalent to the following theorem.

Theorem 3.1 (No-ghost).

$$\mathrm{sign}\, H^0_{\mathrm{rel}}(\mathcal{F}_\varepsilon)(p) = \dim H^0_{\mathrm{rel}}(\mathcal{F}_\varepsilon)(p).$$

Proof. In order to prove the above equality it is most convenient to compare the generating series for dimensions (57) with that for signatures. For the last one to be constructed one may use the Euler–Poincaré principle for signatures (this principle may be proved with the help of positive Hodge–Serre product on the space of the complex [16]), which in the case of string complex amounts to

$$\operatorname{sign} H^0_{\mathrm{rel}}(\mathcal{F}_\varepsilon)(p) = \operatorname{sign} \mathcal{C}^0_{\mathrm{rel}}(\mathcal{F}_\varepsilon)(p) = \operatorname{sign} \mathcal{C}_{\mathrm{rel}}(\mathcal{F}_\varepsilon)(p).$$

The signature of the full complex can be calculated as the weighted trace of Hodge–Serre Hermitian metric operator [16], but it seems that the construction of the generating function for signatures of $\mathcal{C}^0_{\mathrm{rel}}(\mathcal{F}_\varepsilon)(p)$ is more convenient and transparent:

$$\operatorname{sign}_\varepsilon(q) = q^{\frac{1}{2\alpha}p^2+\frac{1}{2}\varrho^2-\varepsilon\frac{d-1}{16}}\, S_\varepsilon(q) \qquad \operatorname{sign} \mathcal{C}^0_{\mathrm{rel}}(\mathcal{F}_\varepsilon)(p) = \operatorname{sign}_\varepsilon(q)|_0. \tag{65}$$

The signatures are multiplicative with respect to the tensor product and consequently the generating function $S_\varepsilon(q)$ is given as the product of the series corresponding to the contributions from appropriately normalized, separate families of modes. First of all, it factors into string and respectively ghost generating series: $S_\varepsilon(q) = S^{\mathrm{str}}_\varepsilon(q)\, S^{\mathrm{gh}}_\varepsilon(q)$.

The string part of the generating function is determined by the Lorenzian character of the metric present in (1) and (2). The states excited by odd number of bosonic or fermionic timelike oscillators have strictly negative 'norm' and the contribution of this timelike sector to the signatures is described by the functions which count the differences between even and odd partitions of the level number. Hence, the total contribution of the string sector is

$$S^{\mathrm{str}}_\varepsilon(q) = 2^{(1-\varepsilon)([\frac{d+2}{2}]-1)}(P_-(q))^d\,(P_{\varepsilon+}(q))^d \prod_{n>0}(1+q^n)^{-1} \prod_{n>0}(1-q^{n-\frac{\varepsilon}{2}}).$$

The first term corresponds to the positive scalar product on the space of Dirac vacuum states, while the two subsequent ones correspond to the spatial/Liouville bosons and fermions respectively.

In order to determine the ghost factor of the generating function it is enough to notice that only homogeneous doublet excitations of $c_{-n}b_{-n}$ and $\gamma_{-r}\beta_{-r}$ give a non-zero contribution to the signature. The fermionic pairs insert +1 if they appear in an even power and -1 when they are excited by an odd number of times. The bosonic ghost pairs always contribute +1. Taking into account their weights one gets

$$S^{\mathrm{gh}}_\varepsilon(q) = \prod_{n>0}(1-q^{2n}) \prod_{n>0}(1-q^{2(n-\frac{\varepsilon}{2})})^{-1}.$$

Hence the generating function (65) for signatures is equal to that of (57) for dimensions. □

The space $\overline{\mathcal{Z}}^0_0(\mathcal{F}_\varepsilon)(p)$ was already identified with the set of physical states (14) of $\mathcal{F}_\varepsilon(p)$. The positivity of the scalar product (64) on the relative cohomologies allows for straightforward identification of the space of exact co-cycles $\mathcal{D}\overline{\mathcal{Z}}^0_0(\mathcal{F}_\varepsilon)(p)$. This in turn makes it possible to establish a strict relation between the space of states constructed within the framework of 'old covariant' formalism and the space of relative cohomologies.

Note, first, that the scalar product $(\ ,\ )_{\mathrm{phys}}$, which is usually constructed ([2] and references therein) on $\mathcal{F}^{\mathrm{phys}}_\varepsilon(p)$ coincides with that induced from the relative complex. To be more precise: for $\mathcal{F}^{\mathrm{phys}}_\varepsilon(p) \ni \varphi \to \mathrm{i}(\varphi) = \varphi \otimes \omega \in \overline{\mathcal{Z}}^0_0(\mathcal{F}_\varepsilon)(p)$ one has $(\varphi, \varphi')_{\mathrm{phys}} = (\mathrm{i}(\varphi), \mathrm{i}(\varphi'))_{\mathrm{rel}}$. This means that the identification (53) is isometric. The no-ghost theorem guarantees that $\Psi^0_0 \in \mathcal{D}\overline{\mathcal{Z}}^0_1(\mathcal{F}_\varepsilon)(p)$ if and only if $(\Psi^0_0, \cdot)_{\mathrm{rel}} = 0$. Therefore, there are no 'negative norm' states in $\mathcal{F}^{\mathrm{phys}}_\varepsilon(p)$ and the space of cohomologically trivial elements in $\overline{\mathcal{Z}}^0_0(\mathcal{F}_\varepsilon)(p)$ coincides with the isometric image of the radical (the set of null vectors) $\mathcal{N}_\varepsilon(p) \subset \mathcal{F}^{\mathrm{phys}}_\varepsilon(p)$. The considerations above can be summarized in the form of the following corollary.

Corollary 3.1 (BRST correspondence).

$$\frac{\mathcal{F}_\varepsilon^{\rm phys}(p)}{\mathcal{N}_\varepsilon(p)} \stackrel{i_*}{\simeq} H^0_{\rm rel}(\mathcal{F}_\varepsilon)(p) \tag{66}$$

where i_* denotes the canonical isometry induced on the quotient spaces.

The correspondence principle allows for construction of the representatives of the relative cohomology classes. The space $\mathcal{F}_\varepsilon^{\rm phys}(p)$ as well as $\overline{\mathcal{Z}}^0_0(\mathcal{F}_\varepsilon)(p)$ can be most conveniently described in terms of DDF operators [32], which are designed to commute with all quantum constraints (11). Their construction is tightly associated with the choice of the light-cone frame $\{e_\pm, e_i\}$ and their domain is restricted by the condition $p^+ \neq 0$ which makes the Fock-type parametrization of the physical states local in the case when the vacuum states are tachyonic or massless. For completeness, the construction of DDF operators is briefly sketched in appendix B.

There are bosonic $\{A^i_m\}_{m\in\mathbb{Z}}$ and fermionic $\{D^i_r\}_{r\in\mathbb{Z}+\frac{\varepsilon}{2}}$ families of the transverse modes $1 \leqslant i \leqslant d-2$. There are two sets of modes corresponding to the physical Liouville excitations $\{U_m\}_{m\in\mathbb{Z}}$ and their fermionic partners $\{T_r\}_{r\in\mathbb{Z}+\frac{\varepsilon}{2}}$ respectively.

The operators above satisfy standard commutation relations:

$$\begin{array}{ll} [A^i_m, A^j_n] = m\,\delta_{n+m}\delta^{ij} & [U_m, U_n] = m\,\delta_{n+m} \\ \{D^i_m, D^i_n\} = \delta_{n+m}\delta^{ij} & \{T_m, T_n\} = \delta_{n+m}. \end{array} \tag{67}$$

There are in addition two families $\{A^-_m\}_{m\in\mathbb{Z}}$ and $\{G^-_r\}_{r\in\mathbb{Z}+\frac{\varepsilon}{2}}$ of longitudinal Brower modes [4], which form a superconformal algebra:

$$[A^-_m, A^-_n] = (m-n)A^-_{m+n} \qquad [A^-_m, G^-_r] = \left(\frac{m}{2} - r\right) G^-_{m+r} \qquad \{G^-_r, G^-_s\} = 2A^-_{r+s} \tag{68}$$

with the central charge equal to zero. The longitudinal operators (anti)commute with transverse and Liouville modes of (67).

It was proved in [2] that

$$\Psi(p) \in \mathcal{F}_\varepsilon^{\rm phys}(p) \Leftrightarrow \Psi(p) = W({\rm DDF})\omega(p) \tag{69}$$

where $W(\cdot)$ denotes any polynomial in DDF creation operators. The elements of $\overline{\mathcal{Z}}^0_0(\mathcal{F}_\varepsilon)(p)$ are obtained by multiplying (69) with appropriate ghost vacuum vector. The DDF state is null, i.e. cohomologically trivial in the BRST picture, if and only if it contains any longitudinal excitation. Hence, the subspace

$$\mathcal{F}_\varepsilon^{\rm lc}(p) = \{\Psi(p);\ A^-_0\ \Psi(p) = 0\} \tag{70}$$

defines a good section of $\overline{\mathcal{Z}}^0_0(\mathcal{F}_\varepsilon)(p)$ over the quotient $H^0_{\rm rel}(\mathcal{F}_\varepsilon)(p)$. This section does not contain the longitudinal excitations and is called the light-cone gauge slice.

It is tempting to define a global picture for the relative cohomology space by fusing the local spaces of (70):

$$H^0_{\rm rel}(\mathcal{F}_\varepsilon) \simeq \int \frac{dp^+}{p^+}\, d^{d-2}\overline{p}\,\mathcal{F}_\varepsilon^{\rm lc}(p) \tag{71}$$

as a direct integral with respect to the Lorentz-invariant measure. The space (71) is well defined only in the case when the ground states are massive—the light-cone coordinates for the momentum are global on the massive shells.

In order to introduce a consistent global picture in the case of tachyonic or light-like ground states, which seem to be generic, one should introduce the appropriate coverings of the shells by the systems of local light-cone coordinates. This in turn implies the necessity to introduce the transition functions between local Fock space pictures, which are defined by local DDF algebras and local light-cone slices (70).

It would seem to be an attractive idea (especially in the context of hadronic interpretation) to think about the non-critical quantum string as a (topologically) confined system, which is only locally visible as the set of free particles of Fock type. That issue is far beyond the scope of this paper, however.

The models under consideration admit a consistent, i.e. Poincaré invariant, truncation of their spectra. (It is sometimes possible to use the argument of the locality of the conformal field theory corresponding to RNS model [7]. Nothing like that can be applied here as there seems to be no CFT in the conventional sense corresponding to the massive string models.) The reduction procedure can be formulated on the level of the relative complex and, in particular, can result in elimination of the tachyonic ground states from the NS spectrum. In that respect it is analogous to the celebrated GSO projection [11] of the critical string. It should be stressed, however, that the truncated spectrum of the massive strings is never supersymmetric.

The original idea of [11] was to project the theory onto the subspaces with definite fermion parity with respect to (6) and (9). The parity projection operators $P^{\pm}_{\mathcal{F}_\varepsilon} = \frac{1}{2}(1 \pm (-1)^{F_\varepsilon})$ of $\mathcal{F}_\varepsilon$ spaces can be in a natural way extended [14] onto the whole of relative complex by putting

$$P^{\pm}_{\varepsilon\,\mathrm{rel}} = \tfrac{1}{2}(1 \pm (-1)^{F_\varepsilon + F_{\mathrm{gh}} + \mathrm{gh}_{\varepsilon\,\mathrm{rel}}}) \qquad (72)$$

where $F_{\mathrm{gh}} = \sum_{n>0}(b_{-n}c_n + c_{-n}b_n)$ and $\mathrm{gh}_{\varepsilon\,\mathrm{rel}}$ denotes the relative ghost number. The last operator is given by (19) with the zero modes suppressed. It is clear that $D_{\mathrm{rel}} P^{\pm}_{\varepsilon\,\mathrm{rel}} = P^{\pm}_{\varepsilon\,\mathrm{rel}} D_{\mathrm{rel}}$ and $\mathcal{C}^{\pm}_{\mathrm{rel}}(\mathcal{F}_\varepsilon)(p) = P^{\pm}_{\varepsilon\,\mathrm{rel}} \mathcal{C}_{\mathrm{rel}}(\mathcal{F}_\varepsilon)(p)$ are the subcomplexes of the relative complex. Their cohomologies can be easily read off those of the full relative complex. They are simply given by GSO projections of (66). The content of the subspaces $H^{0\pm}_{\mathrm{rel}}(\mathcal{F}_\varepsilon)(p) = P^{\pm}_{\varepsilon\,\mathrm{rel}} H^{0}_{\mathrm{rel}}(\mathcal{F}_\varepsilon)(p)$ can be easily described in terms of the local DDF subalgebra of the light-cone gauge slice (70). In the case of vanishing Liouville momentum $\varrho = 0$ the content of GSO projected spaces can be described as follows.

The space $H^{0-}_{\mathrm{rel}}(\mathcal{F}_1)(p)$ is generated from vacuum vector by the polynomials in transverse and Liouville DDF operators of odd fermion number. It is thus tachyon-free and the spectrum begins from a massive vector particle: $m^2_{1/2} = \alpha(1 - \frac{d-1}{8})$. It is worth stressing that the Liouville modes $T_{-1/2}$ and U_{-1} supplement the transverse ones to form a vector representation of a small group $SO(d-1)$ of massive momentum [10].

The complementary space $H^{0+}_{\mathrm{rel}}(\mathcal{F}_1)(p)$ contains a tachyon of depth $m^2_0 = -\alpha\frac{d-1}{8}$. The first admissible excited level is occupied by a massive vector and massive antisymmetric tensor particle: $m^2_1 = \alpha(2 - \frac{d-1}{8})$. In the physical case of four-dimensional spacetime this tensor corresponds to an axial vector and one has, in fact, two vector particles with different properties with respect to spatial reflections. Both are of mass $m^2 = \frac{13}{8}\alpha$.

The subspaces $H^{0\pm}_{\mathrm{rel}}(\mathcal{F}_0)(p)$ of the Ramond sector are naturally isomorphic. The states of $H^{0+}_{\mathrm{rel}}(\mathcal{F}_0)(p)$ are generated by even and odd DDF polynomials acting on Γ^f-even and respectively Γ^f-odd vacuum spinors. In four dimensions the vacuum contains two massless particles of opposite helicities $h = \pm\frac{1}{2}$, while the first excited level is occupied by two spin-$\frac{3}{2}$ particles of mass $m^2_1 = 2\alpha$. Their states are generated by elementary DDF modes out of vacuum spinors of opposite Γ^f-parity.

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Appendix A

This appendix is devoted to a brief presentation of the properties of Clifford modules and a discussion of the light-cone solutions of the Dirac equation.

The complexified Clifford algebras $\mathcal{C}(d+1,1)^{\mathbb{C}}$ (the signature is in fact meaningless) are irreducibly represented as the endomorphism algebras of complex vector spaces of dimension $2^{[\frac{d+2}{2}]}$. For even d, the algebras are simple and the modules are faithful. In odd spacetime dimensions the algebras split into direct sums of simple ideals and one of them is sent to zero under irreducible representation.

The irreducible representation spaces for the real Clifford algebras $\mathcal{C}(d+1,1)$ form a richer category and in the range of interesting dimensions are described in the following table:

$$\begin{array}{cccccccccc} d & = & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ S(d+1,1) & \sim & \mathbb{R}^4 & \mathbb{C}^4 & \mathbb{H}^4 & \mathbb{H}^4 & \mathbb{H}^8 & \mathbb{C}^{16} & \mathbb{R}^{32} & \mathbb{R}^{32} \end{array} \tag{A.1}$$

where $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$ denote the fields of real numbers, complex numbers and the ring of quaternions respectively. The modules (A.1) are not faithful for $d = 5, 9$. Although it is tempting to exploit the full structure of real Clifford modules, this appendix is mainly restricted to the complexified case. To construct the representations which are convenient in the context of this paper one may proceed as follows.

Distinguish two Clifford generators Γ^0, Γ^1 to form the light-cone basis $\Gamma_\pm = \frac{1}{\sqrt{2}}(\Gamma_0 \pm \Gamma_1)$. The remaining basis elements $\Gamma^2, \ldots, \Gamma^{d-1}, \Gamma^l, \Gamma^f$ generate the Euclidean Clifford algebra $\mathcal{C}(d,0)$. Let $S(d,0)$ be the the irreducible representation module for its complexification and let $\{\gamma^a; a = 2, \ldots, d-1, l, f\}$ be the set of endomorphisms representing the generators. Then the generators of the full algebra $\mathcal{C}(d+1,1)$ are represented on $S(d+1,1) \cong S(d,0) \otimes \mathbb{R}^2$ as follows:

$$\begin{aligned} \Gamma^a &\sim \gamma^a \otimes D \\ \Gamma^1 &\sim 1 \otimes A \\ \Gamma^0 &\sim 1 \otimes J \end{aligned} \qquad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.2}$$

Let $\langle , \rangle_{\mathrm{E}}$ denote the positive Hermitian form on $S(d,0)$ such that $(\gamma^a)^+ = \gamma^a$ and put

$$\langle u^{(1)}, u^{(2)} \rangle = \mathrm{i}(\langle u_1^{(1)}, u_2^{(2)} \rangle_{\mathrm{E}} - \langle u_2^{(1)}, u_1^{(2)} \rangle_{\mathrm{E}}) \qquad u^{(i)} = \begin{pmatrix} u_1^{(i)} \\ u_2^{(i)} \end{pmatrix} \in S(d+1,1). \tag{A.3}$$

Then $\langle\, , \rangle$ has all the properties desired for the constructions in the Ramond sector (10).

Using the representation (A.2) one may easily solve the Dirac equation in light-cone coordinates. The complete set $V(p)$ of solutions at fixed momentum is parametrized by $S(d,0)$ and is described by

$$v(p) = \begin{pmatrix} \frac{1}{\sqrt{2}p^+}(\overline{p} + \sqrt{\alpha}\varrho\gamma^l)u \\ u \end{pmatrix} \qquad u \in S(d,0) \quad \overline{p} = \sum_{i=2}^{d} \gamma^i\, p_i. \tag{A.4}$$

Note that the set $V(p)$ is stable with respect to the action of Γ^f as well as with respect to the action of shifted transverse and Liouville zero modes $\hat{d}_0^i, \hat{t}_0$ of (41).

The scalar product (45) introduced on the Ramond relative complex can be expressed on the ground states as the light-like component of the conserved current: $(v(p), v(p))_{\mathrm{rel}} = -\langle v(p), \Gamma^+ v(p) \rangle = (u,u)_{\mathrm{E}}$ and is strictly positive.

Appendix B

The DDF operators are constructed [2] according to the method formulated in [4] and are defined in terms of bosonic and fermionic conformal fields:

$$X^{\mu}(z) = \sqrt{\alpha}x^{\mu} - \mathrm{i}\frac{1}{\sqrt{\alpha}}P^{\mu}\ \log(z) + \sum_{m\neq 0}\frac{\mathrm{i}}{m}a^{\mu}_{m}z^{-m} \qquad P^{\mu}(z) = X'^{\mu}(z)$$

$$\varphi(z) = -\mathrm{i}\varrho\log(z) + \sum_{m\neq 0}\frac{\mathrm{i}}{m}u_{m}z^{-m} \qquad \Pi(z) = \varphi'(z)$$

$$\Psi^{\mu}(z) = \sum_{r\in\mathbb{Z}+\frac{\varepsilon}{2}} d^{\mu}_{r}z^{-r} \qquad \Psi^{l}(z) = \sum_{r\in\mathbb{Z}+\frac{\varepsilon}{2}} t^{\mu}_{r}z^{-r}$$

with standard (anti)commutation rules [33]. The prime $'$ denotes the action of the differential operator $-\mathrm{i}z\frac{\mathrm{d}}{\mathrm{d}z}$.

It is convenient to introduce the rescaled light-cone components of the fields:

$$X_{\pm}(z) \mapsto \left(\frac{\sqrt{\alpha}}{p^{+}}\right)^{\pm 1} X_{\pm}(z) \qquad P_{\pm}(z) \mapsto \left(\frac{\sqrt{\alpha}}{p^{+}}\right)^{\pm 1} P_{\pm}(z)$$

$$\Psi_{\pm}(z) \mapsto \left(\frac{\sqrt{\alpha}}{p^{+}}\right)^{\pm 1} \Psi_{\pm}(z).$$

The transverse DDF operators are given by the formulae

$$A^{i}_{m} = \frac{1}{2\pi\mathrm{i}}\oint\frac{\mathrm{d}z}{z} : (P^{i} - m\Psi^{i}\Psi_{+})\mathrm{e}^{\mathrm{i}mX_{+}} :$$

$$D^{i}_{r} = \frac{1}{2\pi\mathrm{i}}\oint\frac{\mathrm{d}z}{z} : \left(\Psi^{i}P_{+}^{\frac{1}{2}} - P^{i}\Psi_{+}P_{+}^{-\frac{1}{2}} - \frac{\mathrm{i}}{2}\Psi^{i}\Psi_{+}\Psi'_{+}\right)\mathrm{e}^{\mathrm{i}rX_{+}} : .$$

The expressions for Liouville modes are a little more complicated:

$$U_{m} = \frac{1}{2\pi\mathrm{i}}\oint\frac{\mathrm{d}z}{z} : (\Pi - m\Psi^{l}\Psi_{+} + 2\sqrt{\beta}(P_{+}^{-1}(P'_{+} + m\Psi_{+}\Psi'_{+})))\mathrm{e}^{\mathrm{i}mX_{+}} :$$

$$T_{r} = \frac{1}{2\pi\mathrm{i}}\oint\frac{\mathrm{d}z}{z} : \left(\Psi^{l}P_{+}^{\frac{1}{2}} - \Pi\Psi_{+}P_{+}^{-\frac{1}{2}} - \frac{\mathrm{i}}{2}\Psi^{l}\Psi_{+}\Psi'_{+} - 4\sqrt{\beta}(\Psi_{+}P_{+}^{-1})'P_{+}^{\frac{1}{2}}\right)\mathrm{e}^{\mathrm{i}rX_{+}} :$$

while the formulae for longitudinal operators are

$$\tilde{A}^{-}_{m} = \frac{1}{2\pi\mathrm{i}}\oint\frac{\mathrm{d}z}{z} : \left(P_{-} - m\Psi_{-}\Psi_{+} - \frac{\mathrm{i}}{2}P_{+}^{-1}(mP'_{+} + m^{2}\Psi_{+}\Psi'_{+})\right)\mathrm{e}^{\mathrm{i}mX_{+}} :$$

$$\tilde{G}^{-}_{r} = \frac{1}{2\pi\mathrm{i}}\oint\frac{\mathrm{d}z}{z} : \left(\Psi_{-}P_{+}^{\frac{1}{2}} - P_{-}\Psi_{+}P_{+}^{-\frac{1}{2}} - \frac{\mathrm{i}}{2}\Psi_{-}\Psi_{+}\Psi'_{+} + \frac{\varepsilon}{16}\Psi_{+}P_{+}^{-\frac{3}{2}}\right)\mathrm{e}^{\mathrm{i}rX_{+}} :$$
$$+\frac{1}{2\pi\mathrm{i}}\oint\frac{\mathrm{d}z}{z} : \left(\frac{1}{8}(\Psi_{+}P_{+}^{-1})'P_{+}^{-\frac{3}{2}}P'_{+} - \frac{5}{4}(\Psi_{+}P_{+}^{-1})'P_{+}^{\frac{1}{2}}\right.$$
$$\left.-\frac{\mathrm{i}}{8}\Psi_{+}\Psi'_{+}\Psi''_{+}P_{+}^{-\frac{7}{2}}\right)\mathrm{e}^{\mathrm{i}rX_{+}} : .$$

Note that the various powers of P_{+} encode in fact the power series expansions around 1. Only a finite number of terms survive under the integrals when the operators act on the states with a finite number of components.

The longitudinal operators above neither (anti)commute with transverse modes nor with Liouville ones. In order to introduce the diagonal basis one defines

$$A^{-}_{m} = \tilde{A}^{-}_{m} - L^{\mathrm{DDF}}_{m} + \tfrac{1}{2}\delta_{m0} \qquad G^{-}_{r} = \tilde{G}^{-}_{r} - G^{\mathrm{DDF}}_{r}$$

where L^{DDF}_{m} and G^{DDF}_{r} denote the operators given by the standard expressions (11) in transverse and Liouville DDF modes with $a_{\varepsilon} = (\varepsilon - 1)\frac{d-1}{16}$.

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